

Vektorilaskenta

Itä-Suomen yliopisto,
verkkomateriaali



UNIVERSITY OF
EASTERN FINLAND

Määritelmä 5.1

Olkoot $\mathbf{u}$ ja $\mathbf{v}$ avaruuden $\mathbb{R}^3$ vektoreita. Olkoon $\mathbf{w}$ avaruuden $\mathbb{R}^3$ vektori siten, että

- (a) $\mathbf{u} \cdot \mathbf{w} = 0$ ja $\mathbf{v} \cdot \mathbf{w} = 0$,
- (b) $|\mathbf{w}| = |\mathbf{u}||\mathbf{v}| \sin \theta$, missä θ on vektorien $\mathbf{u}$ ja $\mathbf{v}$ välinen kulma,
- (c) järjestetty kolmikko $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ on positiivisesti suunnistettu (oikean käden systeemi).

Tällöin $\mathbf{w}$ on vektorien $\mathbf{u}$ ja $\mathbf{v}$ **ristitulo** ja merkitään $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

Kuva tähän.

Lause 5.2

Olkoot $\mathbf{u}, \mathbf{v}$ ja $\mathbf{w}$ vektoreita avaruudessa $\mathbb{R}^3$ ja t skalaari. Tällöin on voimassa

- (a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}),$
- (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$
- (c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w},$
- (d) $t(\mathbf{u} \times \mathbf{v}) = (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}),$
- (e) $\mathbf{u} \times \mathbf{u} = \mathbf{0},$
- (f) $\mathbf{0} \times \mathbf{u} = \mathbf{0},$
- (g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$
- (h) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u},$
- (i) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{u}$

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(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w},$

(d) $t(\mathbf{u} \times \mathbf{v}) = (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}),$

(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0},$

(f) $\mathbf{0} \times \mathbf{u} = \mathbf{0},$

(g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$ Lagrange kolmitulokaava

(h) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u},$

(i) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{u}$

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Todistus. Todistetaan kohta (g).

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi $\mathbf{i}$ -komponenttia.

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1$$

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi $\mathbf{i}$ -komponenttia.

$$\begin{aligned} & (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\ &= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \end{aligned}$$

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi i -komponenttia.

$$\begin{aligned} & (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\ &= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\ &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \end{aligned}$$

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$$\begin{aligned} & (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\ &= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\ &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) \\ &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) + (v_1 u_1 w_1 - w_1 u_1 v_1) \end{aligned}$$

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$$\begin{aligned} & (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\ &= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\ &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) \\ &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) + (v_1 u_1 w_1 - w_1 u_1 v_1) \\ &= v_1(u_1 v_1 + u_2 v_2 + u_3 w_3) - w_1(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi $\mathbf{i}$ -komponenttia.

$$\begin{aligned}
 & (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\
 &= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\
 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\
 &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) \\
 &= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) + (v_1 u_1 w_1 - w_1 u_1 v_1) \\
 &= v_1(u_1 v_1 + u_2 v_2 + u_3 w_3) - w_1(u_1 v_1 + u_2 v_2 + u_3 v_3) \\
 &= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})
 \end{aligned}$$

Siis

$$\begin{array}{rcl}
 (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \mathbf{i} & = & [v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})] \mathbf{i} \\
 + (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_2 \mathbf{j} & = & [v_2(\mathbf{u} \cdot \mathbf{w}) - w_2(\mathbf{u} \cdot \mathbf{v})] \mathbf{j} \\
 + (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_3 \mathbf{k} & = & [v_3(\mathbf{u} \cdot \mathbf{w}) - w_3(\mathbf{u} \cdot \mathbf{v})] \mathbf{k} \\
 \hline
 \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & = & \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}).
 \end{array}$$



Seuraus 5.3 (Jakobin identiteetti)

Jos $\mathbf{u}, \mathbf{v}$ ja $\mathbf{w}$ ovat vektoreita avaruudessa $\mathbb{R}^3$, niin

$$\mathbf{r} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

Todistus 5.4

Lagrangen kolmitulokaavan perusteella

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

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$$\begin{array}{rcl} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & = & \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \\ + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) & = & \mathbf{w}(\mathbf{v} \cdot \mathbf{w}) - \mathbf{u}(\mathbf{v} \cdot \mathbf{v}) \\ + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) & = & \mathbf{u}(\mathbf{w} \cdot \mathbf{w}) - \mathbf{v}(\mathbf{w} \cdot \mathbf{v}) \\ \hline \mathbf{r} & = & \end{array}$$

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$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \cancel{\mathbf{v}(\mathbf{u} \cdot \mathbf{w})} - \cancel{\mathbf{w}(\mathbf{u} \cdot \mathbf{v})} \\ + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) &= \cancel{\mathbf{w}(\mathbf{v} \cdot \mathbf{u})} - \cancel{\mathbf{u}(\mathbf{v} \cdot \mathbf{w})} \\ + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= \cancel{\mathbf{u}(\mathbf{w} \cdot \mathbf{v})} - \cancel{\mathbf{v}(\mathbf{w} \cdot \mathbf{u})} \\ \hline \mathbf{r} &= \end{aligned}$$

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