Vektorilaskenta

Itä-Suomen yliopisto,
verkkomateriaali

UNIVERSITY OF EASTERN FINLAND

Määritelmä

Olkoot $\mathbf{u}$ ja $\mathbf{v}$ avaruuden $\mathbb{R}^3$ vektoreita. Olkoon $\mathbf{w}$ avaruuden $\mathbb{R}^3$ vektori siten, että

(a) $\mathbf{u} \cdot \mathbf{w} = 0$ ja $\mathbf{v} \cdot \mathbf{w} = 0$,

(b) $|\mathbf{w}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, missä θ on vektorien $\mathbf{u}$ ja $\mathbf{v}$ välinen kulma,

(c) järjestetty kolmikko $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ on positiivisesti suunnistettu (oikean käden systeemi).

Tällöin $\mathbf{w}$ on vektorien $\mathbf{u}$ ja $\mathbf{v}$ ristitulo ja merkitään $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

Kuva tähän.

Lause

Olkoot $\mathbf{u}, \mathbf{v}$ *ja* $\mathbf{w}$ *vektoreita avaruudessa* $\mathbb{R}^3$ *ja* t *skalaari. Tällöin on voimassa*

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$,

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$,

(d) $t(\mathbf{u} \times \mathbf{v}) = (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v})$,

(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,

(f) $\mathbf{0} \times \mathbf{u} = \mathbf{0}$,

(g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$,

(h) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$,

(i) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{u}$

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(g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$, Lagrangen kolmitulokaava

(h) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$,

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Todistus. Todistetaan kohta (g).

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi **i**-komponenttia.

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1$$

Lagrangen kolmitulokaava

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Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi $\mathbf{i}$-komponenttia.

$$
\begin{aligned}
&(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\
&= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2
\end{aligned}
$$

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&= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\
&= u_2\big(v_1 w_2 - v_2 w_1\big) - u_3\big(v_3 w_1 - v_1 w_3\big)
\end{aligned}
$$

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&= u_2\big(v_1 w_2 - v_2 w_1\big) - u_3\big(v_3 w_1 - v_1 w_3\big) \\
&= v_1\big(u_2 v_2 + u_3 w_3\big) - w_1\big(u_2 v_2 + u_3 v_3\big)
\end{aligned}
$$

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&= u_2\big(v_1 w_2 - v_2 w_1\big) - u_3\big(v_3 w_1 - v_1 w_3\big) \\
&= v_1\big(u_2 v_2 + u_3 w_3\big) - w_1\big(u_2 v_2 + u_3 v_3\big) \\
&= v_1\big(u_2 v_2 + u_3 w_3\big) - w_1\big(u_2 v_2 + u_3 v_3\big) + \big(v_1 u_1 w_1 - w_1 u_1 v_1\big)
\end{aligned}
$$

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&= v_1\big(u_2 v_2 + u_3 w_3\big) - w_1\big(u_2 v_2 + u_3 v_3\big) + \big(v_1 u_1 w_1 - w_1 u_1 v_1\big) \\
&= v_1\big(u_1 v_1 + u_2 v_2 + u_3 w_3\big) - w_1\big(u_1 v_1 + u_2 v_2 + u_3 v_3\big)
\end{aligned}
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Lagrangen kolmitulokaava

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi **i**-komponenttia.

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\begin{aligned}
&(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\
&= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\
&= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\
&= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) \\
&= v_1(u_2 v_2 + u_3 w_3) - w_1(u_2 v_2 + u_3 v_3) + (v_1 u_1 w_1 - w_1 u_1 v_1) \\
&= v_1(u_1 v_1 + u_2 v_2 + u_3 w_3) - w_1(u_1 v_1 + u_2 v_2 + u_3 v_3) \\
&= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})
\end{aligned}
$$

Lagrangen kolmitulokaava

Tarkastellaan vektorin $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ komponenteista aluksi $\mathbf{i}$-komponenttia.

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&(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 \\
&= u_2(\mathbf{v} \times \mathbf{w})_3 - u_3(\mathbf{v} \times \mathbf{w})_2 \\
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&= v_1\big(u_2 v_2 + u_3 w_3\big) - w_1\big(u_2 v_2 + u_3 v_3\big) \\
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&= v_1\big(u_1 v_1 + u_2 v_2 + u_3 w_3\big) - w_1\big(u_1 v_1 + u_2 v_2 + u_3 v_3\big) \\
&= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})
\end{aligned}$$

Siis

$$\begin{array}{rl}
(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1\mathbf{i} & = [v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})]\mathbf{i} \\
+(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_2\mathbf{j} & = [v_2(\mathbf{u} \cdot \mathbf{w}) - w_2(\mathbf{u} \cdot \mathbf{v})]\mathbf{j} \\
+(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_3\mathbf{k} & = [v_3(\mathbf{u} \cdot \mathbf{w}) - w_3(\mathbf{u} \cdot \mathbf{v})]\mathbf{k} \\
\hline
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}).
\end{array}$$

□

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Seuraus (Jakobin identiteetti)

Jos $\mathbf{u}, \mathbf{v}$ *ja* $\mathbf{w}$ *ovat vektoreita avaruudessa* $\mathbb{R}^3$*, niin*

$$\mathbf{r} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

Todistus

Lagrangen kolmitulokaavan perusteella

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

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$$\begin{array}{rl} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \\ +\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) & = \mathbf{w}(\mathbf{v} \cdot \mathbf{w}) - \mathbf{u}(\mathbf{v} \cdot \mathbf{v}) \\ +\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) & = \mathbf{u}(\mathbf{w} \cdot \mathbf{w}) - \mathbf{v}(\mathbf{w} \cdot \mathbf{v}) \\ \hline \mathbf{r} & = \end{array}$$

Lagrangen kolmitulokaava

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$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

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$$\begin{array}{rl} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & = \cancel{\mathbf{v}(\mathbf{u} \cdot \mathbf{w})} - \cancel{\mathbf{w}(\mathbf{u} \cdot \mathbf{v})} \\ +\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) & = \cancel{\mathbf{w}(\mathbf{v} \cdot \mathbf{w})} - \cancel{\mathbf{u}(\mathbf{v} \cdot \mathbf{v})} \\ +\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) & = \cancel{\mathbf{u}(\mathbf{w} \cdot \mathbf{w})} - \cancel{\mathbf{v}(\mathbf{w} \cdot \mathbf{v})} \\ \hline \mathbf{r} & = \end{array}$$

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