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On injectivity radius in configuration space and in moduli space

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ABSTRACT. We shall estimate the injectivity radius in the configuration space and in the moduli space in terms of the hyperbolic geometry.

1. Introduction

Let n be an integer greater than three. We consider the space of ordered n points of the Riemann sphere $\hat{\mathbb{C}}$ modulo the action of Möbius transformations. We call the space the *configuration space* of n points and denote it by $\mathcal{M}_n$ (See §2 for the precise definition). The configuration space $\mathcal{M}_n$ is obtained from the Teichmüller space of the Riemann sphere with n punctures like the moduli space. Both spaces, the configuration space and the moduli space, are endowed with a natural distance, the Teichmüller distance. We are interested in the geometry of both spaces with respect to the Teichmüller distance. Especially, we focus on the injectivity radius in those spaces.

Let (M, d) be a metric space. The injectivity radius at $p \in M$ is the shortest length of non-trivial closed curves passing through p. In this paper, we shall estimate the injectivity radius at a point in the configuration space and in the moduli space in terms of the hyperbolic geometry of the Riemann surface for the point.

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2. Preliminaries and main results

2.1. Moduli space and configuration space. Let X_0 be a Riemann surface of type (g, n), that is, X_0 is a Riemann surface of genus g with n punctures. We always assume that $2g - 2 + n > 0$. Hence, X_0 admits the hyperboli metric.

We consider a pair (X, f) of a Riemann surface X and a quasiconformal mapping f from X_0 onto X. Two such pairs (X_i, f_i) $(i = 1, 2)$ are equivalent if there exists a conformal mapping $h : X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. We denote by $[X, f]$ the equivalence class represented by (X, f). The set of all equivalence classes $[X, f]$ is called the *Teichmüller space* of X_0 and it is denoted by $T(X_0)$.

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For any points $[X_i, f_i]$ $(i = 1, 2)$, we define the Teichmüller distance between them by

$$d_T([X_1, f_1], [X_2, f_2]) = \frac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal mapping from X_1 to X_2 homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of f. It is known that the Teichmüller space is a complex manifold with dimension $3g - 3 + n$ and the Teichmüller distance coincides with the Kobayashi distance. It is also known that the mapping class group $Mod(X_0)$, the group of homotopy classes of quasiconformal selfmaps of X_0, is the biholomorphic automorphisms of $T(X_0)$. In fact, a mapping class χ_ϕ of a quasiconformal selfmap ϕ of X_0 acts on $T(X_0)$ by

$$\chi_\phi([X, f]) = [X, f \circ \phi^{-1}].$$

It is easy to see that the action is well-defined and χ_ϕ is isometric with respect to the Teichmüller distance. Moreover, $Mod(X_0)$ acts on $T(X_0)$ properly discontinuously and the quotient space $M(X_0) := T(X_0)/Mod(X_0)$ which is called the *moduli space* is a complex orbifold with dimension $3g - 3 + n$. Since $Mod(X_0)$ is the isometry group of $T(X_0)$, the Teichmüller distance is projected to the moduli space. We use the same symbol d_T as the projected distance on $M(X_0)$.

For $n \geq 4$, we consider ordered n-tuples $(z_1, z_2, \dots z_n)$ of distinct points of $\hat{\mathbb{C}}$. Such two n-tuples $(z_1, z_2, \dots, z_n)$ and $(w_1, w_2, \dots, w_n)$ are equivalent if there exists a Möbius transformation φ such that $\varphi(z_i) = w_i$ $(i = 1, 2, \dots, n)$. The configuration space $\mathcal{M}_n$ is the set of all equivalence classes.

Since there exists a Möbius transformation φ such that $\varphi(z_{n-2}) = 0, \varphi(z_{n-1}) = 1$ and $\varphi(z_n) = \infty$, the space $\mathcal{M}_n$ is identified with the set of ordered $(n-3)$-tuples $(z_1, z_2, \dots z_{n-3})$ of distinct points in $\mathbb{C} \setminus \{0, 1\}$. In this paper, we use this identification for $\mathcal{M}_n$.

The configuration space $\mathcal{M}_n$ is endowed with a natural distance, the Teichmüller distance $d_{T,n}$. For any two points $p_j = (z_1^j, z_2^j, \dots z_{n-3}^j)$ $(j = 1, 2)$, the Teichmüller distance between them is defined by

$$d_{T,n}(p_1, p_2) = \frac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal self mappings f of $\hat{\mathbb{C}}$ with $f(0) = 0, f(1) = 1$ and $f(z_i^1) = z_i^2$ $(i = 1, 2, \dots, n-3)$, and $K(f)$ is the maximal dilatation of f.

2.2. Mapping class group. Here, we present the Bers-Thurston classification of mapping classes.

Let X_0 be a Riemann surface of finite type and $\phi : X_0 \to X_0$ a quasiconformal selfmap of X_0. We say that ϕ is *reduced* if there exists a finite number of non-trivial simple close curves $c_1, c_2, \dots, c_k$ on X_0 satisfying;

(1) $c_i \cap c_j = \emptyset$ if $i \neq j$;
(2) each c_i is not homotopic to a puncture of X_0;
(3) $\phi(c_i)$ is equal to some c_j $(i = 1, 2, \dots, k)$.

If ϕ is reduced for $C = \{c_1, c_2, \dots, c_k\}$ as above, the mapping ϕ defines a permutation of the set of connected components of $X_0 \setminus \{c_1, c_2, \dots, c_k\}$. We denote by S_j $(j = 1, 2, \dots, \ell)$ such a connected component. Then, there exists $m \in \mathbb{N}$ such that ϕ^m fixes every S_j. We take m as the minimal number with this property. We call

ϕ^m the component map of ϕ. A reduced map is called *completely reduced* if the component map ϕ^m is irreducible on each component S_j. A quasiconformal map $\phi : X_0 \to X_0$ is called *reducible* if it is homotopic to a reduced mapping, and it is called *irreducible* if it is not homotopic to a reducible mapping. It is known that every reducible mapping is homotopic to a completely reduced mapping. Hereafter, we suppose that ϕ is completely reduced if it is reducible.

For a quasiconformal self mapping ϕ of X_0, we denote by χ_ϕ the mapping class represented by ϕ. Then we have the following classification;

DEFINITION 2.1. For $\chi_\phi \in Mod(X_0)$,

(1) it is called *elliptic* if it is of finite order;
(2) it is called *parabolic* if it is of infinite order and the component map ϕ^m is homotopic to a mapping of finite order on every S_j;
(3) it is called *pseudo-hyperbolic* if ϕ is reducible but χ_ϕ is neither parabolic nor elliptic;
(4) it is called *hyperbolic* if it is of infinite order and ϕ is irreducible.

For $\chi \in Mod(X_0)$, we define

$$a(\chi) = \inf_{p \in T(X_0)} d_T(p, \chi(p)).$$

Then, the following is known [**1**];

PROPOSITION 2.1. *Let χ be a mapping class of X_0. Then, the following hold;*

(1) *χ is elliptic if $\chi(p) = p$ for some $p \in T(X_0)$;*
(2) *χ is parabolic if $a(\chi) = 0$ but $d_T(p, \chi(p)) > 0$ for any $p \in T(X_0)$;*
(3) *χ is pseudo-hyperbolic if $a(\chi) > 0$ but $d_T(p, \chi(p)) < a(\chi)$ for any $p \in T(X_0)$;*
(4) *χ is hyperbolic if $a(\chi) > 0$ and $a(\chi) = d_T(p, \chi(p))$ for some $p \in T(X_0)$.*

2.3. Main results. We consider the injectivity radius in the moduli space $M(X_0)$ and the configuration space $\mathcal{M}_n$.

The injectivity radius $r_p(\mathcal{M}_n)$ of $\mathcal{M}_n$ at $p \in \mathcal{M}_n$ is the smallest length of non-trivial curves passing through p in $\mathcal{M}_n$ with respect to the Teichmüller distance. On the other hand, since the moduli space $M(X_0)$ is not a manifold, we define the injectivity radius on $M(X_0)$ by using the Teichmüller space.

Let $\Pi : T(X_0) \to M(X_0)$ be the canonical projection. For $p \in M(X_0)$, we define the injectivity radius $r_p(M(X_0))$ of $M(X_0)$ at p by

$$r_p(M(X_0)) = \inf_{\chi_\phi \in Mod(X_0)' \setminus \{id.\}} d_T(P, \chi_\phi(P)),$$

where $Mod(X_0)'$ is the set of non-elliptic elements of $Mod(X_0)$ and P is a point in $T(X_0)$ with $\Pi(P) = p$. Noting that

$$d_T(\chi_\psi(P), \chi_\phi(\chi_\psi(P))) = d_T(P, \chi_\psi^{-1}\chi_\phi\chi_\psi(P)),$$

we verify that the above definition does not depend on the point P in $\Pi^{-1}(p)$.

Before stating our theorems, we give a related result on the injectivity radius in the configuration space by Yamanoi [**6**].

For $p = (z_1, z_2, \ldots, z_{n-3}) \in \mathcal{M}_n$, we set $X(p) := \mathbb{C} \setminus \{0, 1, z_1, z_2, \ldots, z_{n-3}\}$. Yamanoi recently shows the following result which gives an estimate of $r_p(\mathcal{M}_n)$ from below in terms of the hyperbolic structure of $X(p)$.

THEOREM (Yamanoi). *For any* $p \in \mathcal{M}_n$,

$$\frac{\mathcal{E}_p}{50n} \leq r_p(\mathcal{M}_n) \tag{2.1}$$

holds, if there is no essential annulus A *in* $X(p)$ *with* $Mod(A) \geq -\frac{1}{2\pi}\log \mathcal{E}_p$

In [**6**], Yamanoi uses the above theorem to show that the Gol'dberg conjecture in the Nevanlinna theory is true.

In this paper, we shall show the following;

THEOREM 2.1. *For any* $p \in \mathcal{M}_n$,

$$\min\left\{\log(2+\sqrt{5}), \log\sqrt{\left(\frac{\ell_p}{\pi}\right)^2+1}\right\} \leq r_p(\mathcal{M}_n) \tag{2.2}$$

holds, where ℓ_p *is the length of the shortest closed geodesic in* $X(p)$.

REMARK 2.1. Theorem 2.1 has an advantage than Yamanoi's theorem since the injectivity radius is estimated from below by a quatity independent of n. However, the following result on the injectivity radius in the moduli space needs a quantity which depends on the type of the Riemann surface X_0.

THEOREM 2.2. *Let* X_0 *be a Riemann surface of type* (g,n) *with* $2g-2+n>0$. *Then, for any* $p=[X_p,f_p] \in M(X_0)$, *we have*

$$M(g,n)^{-1}\min\left\{\log 2, \log\left(\frac{\ell_p^2}{\pi^2}+1\right)\right\} \leq r_P(M(X_0)), \tag{2.3}$$

where $M(g,n)=\{84(g-1)+4n\}(2g-2+n)!$ *and* ℓ_p *is the length of the shortest closed geodesics in* X_p.

3. Proof of Theorem 2.1

Take a base point $p_0 = (z_1^0, z_2^0, \ldots, z_{n-3}^0) \in \mathcal{M}_n$ and fix it. We consider the Teichmüller space $T(X(p_0))$ of $X(p_0)$. Let $PMod(X(p_0))$ denote a subgroup of $Mod(X(p_0))$ consisting of mapping classes whose representatives fix each z_i^0 $(i=1,2,\ldots,n-3)$ and $0,1,\infty$. We call it the *pure mapping class group* of $X(p_0)$. The configuration space $\mathcal{M}_n$ is described by the pure mapping class group $PMod(X(p_0))$.

PROPOSITION 3.1. *The configuration space* $\mathcal{M}_n$ *is identified with the quotient space* $T(X(p_0))/PMod(X(p_0))$.

PROOF. We define a map $\pi : T(X(p_0)) \to \mathcal{M}_n$ by

$$\pi([X,w]) = (w(z_1^0), w(z_2^0), \ldots, w(z_{n-3}^0)).$$

Since w fixes $0,1$ and ∞, the mapping π is well-defined. Also, it is easily seen that it is surjective and $\pi([X,w]) = \pi(\chi([X,w]))$ for any $\chi \in PMod(X(p_0))$.

Suppose that $\pi([X_1,w_1]) = \pi([X_2,w_2])$. Then, $w_1(z_i^0) = w_2(z_i^0)$ $(i=1,2,\ldots,n-3)$ and the mapping class χ of $w_1^{-1} \circ w_2$ belongs to $PMod(X(p_0))$. Since $\chi([X_2,w_2]) = [X_1,w_1]$, we conclude that $\mathcal{M}_n = T(X(p_0))/PMod(X(p_0))$. □

From the above proposition, immediately we have;

COROLLARY 3.1. *For any $p \in \mathcal{M}_n$, we have*

$$r_p(\mathcal{M}_n) = \inf_{\chi \in PMod(X(p_0)) \setminus \{id.\}} d_T(P, \chi(P)), \tag{3.1}$$

where $P \in T(X(p_0))$ is a point of $\pi^{-1}(p)$.

Let $p = (z_1, z_2, \ldots, z_{n-3})$ be a point in $\mathcal{M}_n$. For $P = [X_p, w] \in \pi^{-1}(p) \subset T(X(p_0))$ and for $\chi_\phi \in PMod(X(p_0)) \setminus \{id.\}$, we consider $d_T(P, \chi_\phi(P))$.

First all of all, we see that χ_ϕ is not elliptic. Indeed, if χ_ϕ could be elliptic, then there would exist a point $Q \in T(X(p_0))$ such that $\chi_\phi(Q) = Q$. Then, ϕ is regarded as a conformal mapping on the surface of Q, which is a Möbius transformation. However, χ_ϕ is a pure mapping class. Hence, the Möbius transformation must be the identity and $\chi_\phi = id$. It is a contradiction.

Suppose that χ_ϕ is hyperbolic. Song [**5**] shows;

PROPOSITION 3.2. *Let $\phi : X(p_0) \to X(p_0)$ gives a hyperbolic pure mapping class. Then*

$$K(\phi) \geq 2 + \sqrt{5}. \tag{3.2}$$

Hence, if χ_ϕ is hyperbolic, then we have

$$d_T(P, \chi_\phi(P)) \geq \log(2 + \sqrt{5}). \tag{3.3}$$

Next, we suppose that χ_ϕ is of infinite order and that ϕ is completely reduced. Then there exist mutually disjoint non-trivial simple closed curves $c_1, c_2, \ldots, c_k$ on $X(p_0)$ such that $\phi(c_i) = c_j$ for some j and c_i does not bound a puncture $(i = 1, 2, \ldots, k)$. Suppose that $c_i \neq c_j$. Since $c_i \cap c_j = \emptyset$, the sets of punctures bounded by c_i and by c_j are different from each other. It is absurd because ϕ fixes each puncture of X_0. Therefore, we conclude that $\phi(c_i) = c_i$ $(i = 1, 2, \ldots, n-3)$. Since ϕ determines a pure mapping class, we see that the mapping ϕ fixes every connected component of $X(p_0) \setminus \{c_1, c_2, \ldots, c_k\}$ as well as every c_i. This implies that ϕ should be a composition of a product of Dehn twists about $c_1, c_2, \ldots, c_k$ and a self-map of finite order in each component of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$ when χ_ϕ is parabolic. However, the mapping ϕ is homotopic to the identity in each componet of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$ because χ_ϕ is a pure mapping class and ϕ fixes every c_i $(i = 1, 2, \ldots, k)$. Therefore, we conclude that ϕ is a product of Dehn twists abount $c_1, c_2, \ldots, c_k$.

Now, we present an estimate of the maximal dilatations of Dehn twists. Let X be a hyperbolic Riemann surface possibly of infinite type. For a non-trivial simple closed curve $c \subset X$ which is not homotopic to a puncture of X, we denote by $\ell_X(c)$ the hyperbolic length of the geodesic homotopic to c. We denote by $\tau_X(c)$ the Dehn twist about c on X. Then, Matsuzaki [**3**] shows;

PROPOSITION 3.3. *Let $c_1, c_2, \ldots$ be mutually disjoint simple closed geodesics on X. If a quasiconformal self mapping f of X is homotopic to a product of Dehn twists $\prod_{i=1}^{\infty} \tau_X(c_i)^{n_i}$ $(n_i \in \mathbb{Z} \setminus \{0\})$, then*

$$K(f) \geq \sup_i \left\{ \left(\frac{(2|n_i| - 1)\ell_X(c_i)}{\pi} \right)^2 + 1 \right\}^{1/2}. \tag{3.4}$$

Therefore, if χ_ϕ is parabolic, then we have

$$d_T(P, \chi_\phi(P)) \geq \sqrt{\left(\frac{\ell_p}{\pi}\right)^2 + 1}. \tag{3.5}$$

Finally, we suppose that χ_ϕ is pseudo-hyperbolic. Then, $a(\chi_\phi) > 0$ and we have a sequence $\{P_n\}_{n=1}^{\infty}$ of $T(X(p_0))$ such that

$$a(\chi_\phi) < d_T(P_n, \chi_\phi(P_n))$$

and

$$\lim_{n\to\infty} d_T(P_n, \chi_\phi(P_n)) = a(\chi_\phi).$$

In fact, the sequence "converges" to a boundary point of the Teichmüller space. Here, we consider a Riemann surface $\hat{X}_0$ with nodes which is obtained from X_0 by squeezing each c_i to a node. It is not hard to see that the mapping $\phi : X_0 \to X_0$ is projected to a mapping $\hat{\phi} : \hat{X}_0 \to \hat{X}_0$ which keeps every node fixed. It is known that the number $a(\chi_\phi)$ is obtained from the maximal dilatation of $\hat{\phi}$ on $\hat{X}_0$.

Since χ_ϕ is pseudo-hyperbolic, there exists a set of connected components of $\hat{X}_0 \setminus \{\text{nodes}\}$, say $\mathcal{S}$, such that $\hat{\phi}|\hat{S} : \hat{S} \to \hat{S}$ is irreducible and of infinite order for any $\hat{S} \in \mathcal{S}$. Noting that X_0 is of type $(0, n)$, we see that $\hat{S}$ is also of type $(0, s)$ for some $s \in \mathbb{N}$. The mapping ϕ determines a pure mapping class and fixes every c_i. Hence, $\hat{\phi}|\hat{S}$ also determines a pure mapping class in $\hat{S}$. Therefore, we may use the theorem of Song, that is, we have an estimate;

$$\log K(\hat{\phi}|\hat{S}) \geq \log(2 + \sqrt{5}).$$

Thus, we obtain

$$a(\chi_\phi) \geq \inf_{\hat{S}} \frac{1}{2} \log K(\hat{\phi}|\hat{S}) \geq \inf_{\hat{S}} \frac{1}{2} \log(2 + \sqrt{5}). \tag{3.6}$$

From (3.2), (3.5) and (3.6), we have the desired result.

4. Proof of Theorem 2.2

If $\chi \in Mod(X_0)'$ is hyperbolic, then it follows from a theorem of Penner [**4**] that

$$d_T(p, \chi(p)) \geq \frac{\log 2}{12g - 12 + 4n}. \tag{4.1}$$

If $\chi := \chi_\phi$ is pseudo-hyperbolic, then there exist mutually disjoint simple closed curves $c_1, c_2, \ldots, c_k$ on X_0 and $m \in \mathbb{N}$ such that

(1) ϕ is completely reduced for $c_1, c_2, \ldots, c_k$;
(2) ϕ^m keeps every connected component of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$ fixed;
(3) the number m is the smallest one with the above property.

Then, we see that for some component S of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$, $\phi^m|S : S \to S$ is irreducible and of infinite order. Therefore, by using the same argument as in §3 and the theorem in [**4**] again, we verify that

$$d_T(p, \chi_\phi^m(p)) \geq a(\chi_\phi^m) \geq \frac{\log 2}{12g - 12 + 4n}. \tag{4.2}$$

Since the number of components of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$ is not greater than $2g-2+n$ and ϕ gives a permutation of the set of components, we have $m \leq (2g-2+n)!$. On the other hand,

$$d_T(p, \chi_\phi^m(p)) \leq \sum_{j=1}^{m} d_T(\chi_\phi^{j-1}(p), \chi_\phi^j(p)) \leq m d_T(p, \chi_\phi(p))$$

because χ_ϕ is an isometry with respect to the Teichmüller distance. Therefore, we have

$$d_T(p, \chi_\phi(p)) \geq \frac{\log 2}{4(3g-3+n)(2g-2+n)!}. \tag{4.3}$$

Finally, we suppose that χ_ϕ is parabolic. Then, we may take $m \in \mathbb{N}$ and mutually disjoint simple closed curves $c_1, c_2, \ldots, c_k$ in X_0 as in the previous argument while $\phi^m|S : S \to S$ is homotopic to a homeomorphism of finite order on every connected component S of $X_0 \setminus \{c_1, c_2, \ldots, c_k\}$. Thus, $\phi^{mm'}$ is a product of Dehn twists about $c_1, c_2, \ldots, c_k$ for some $m' \in \mathbb{N}$. Since the Riemann surface X_0 is of type (g, n), the order m' should be less than $84(g-1)+n$.

From [**3**], we conclude that

$$d_T(p, \chi_\phi^{mm'}(p)) \geq \frac{1}{2} \log \left(\frac{\ell_p^2}{\pi^2} + 1 \right).$$

By the same argument as above, we have

$$d_T(p, \chi_\phi(p)) \geq m(g,n)^{-1} \log \left(\frac{\ell_p^2}{\pi^2} + 1 \right), \tag{4.4}$$

where $m(g,n) = 2\{84(g-1)+n\}(2g-2+n)!$.

From (4.1), (4.3) and (4.4), we have the desired result.

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