Notes on Yamanoi's paper "Zeros of higher derivatives of meromorphic functions in the complex plane"

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CHAPTER 1

Introduction

In these notes we consider the paper of Yamanoi [**19**].

1.1. Notation

We denote the Riemann sphere by $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$. The *spherical area form* ω on $\mathbb{P}^1$ is defined, as in [**2**], by

$$\omega = \frac{\sqrt{-1}}{2} \frac{dz \wedge d\overline{z}}{(1+|z|^2)^2},$$

where z is a local coordinate. The name spherical area form stems from the fact that if the set D satisfies $D \subset \mathbb{P}^1$, then the integral $\int_D \omega$ is proportional to the area of D considered as a subset of the sphere $\mathbb{P}^1 \subset \mathbb{R}^3$. The form ω is normalized[1] so that

$$\int_{\mathbb{P}^1} \omega = \int_0^{2\pi} \int_0^\infty \frac{r dr d\theta}{(1+r^2)^2} = \pi \int_0^\infty \frac{2r}{(1+r^2)^2} dr = \pi.$$

Let f be a meromorphic function in the complex plane, and define

$$\mathbb{C}(t) = \{z \in \mathbb{C} : |z| < t\}.$$

Then the *spherical characteristic function* $T(r,f)$ is defined by

$$T(r,f) = \frac{1}{\pi} \int_1^r A(t,f) \frac{dt}{t},$$

with

$$A(t,f) = \int_{\mathbb{C}(t)} f^*\omega = \int_{\mathbb{C}(t)} (f^\#(z))^2 \, dm(z),$$

where $f^*\omega$ is the *pull back* of ω by f and $f^\# = |f'|/(1+|f|^2)$. Hence, $A(t,f)$ is the area of $f(\mathbb{C}(t))$ on the Riemann sphere, counting multiplicity.

For $a \in \mathbb{P}^1 \setminus \{\infty\}$ the counting function $N(r,a,f)$ is defined by

$$N(r,a,f) = \int_1^r n(t,a,f) \frac{dt}{t},$$

where

$$n(t,a,f) = \sum_{z \in D(t)} \operatorname{ord}_z^+(f-a)$$

[1]Hence, in our case, the radius of the Riemann sphere is $R = 1/2$. Surface area is $4\pi R^2 = 4\pi(1/2)^2 = \pi$.

is the number of zeros of $f(z) - a$ on $D(t)$ counting multiplicity. The *reduced counting function* $\overline{N}(r, a, f)$ is similarly defined by

$$\overline{N}(r, a, f) = \int_1^r \overline{n}(t, a, f) \frac{dt}{t},$$

where

$$\overline{n}(t, a, f) = \sum_{z \in D(t)} \min\{1, \mathrm{ord}_z^+(f - a)\}$$

is the number of zeros of $f(z) = a$ on $D(t)$ ignoring multiplicity. If $a = \infty$, then we define

$$n(t, \infty, f) = \sum_{z \in D(t)} \mathrm{ord}_z^- f$$

and the rest of the counting functions in an analogous way. We also define

$$N_1(r, a, f) = N(r, a, f) - \overline{N}(r, a, f).$$

The *chordal distance* between $a, b \in \mathbb{P}^1$ is defined by

$$[a, b] = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}}$$

if $a, b \in \mathbb{P}^1 \setminus \{\infty\}$, and by

$$[a, \infty] = \frac{1}{\sqrt{1 + |a|^2}}$$

if[2] $b = \infty$. Then the *proximity function* $m(r, a, f)$ is defined by

$$m(r, a, f) = \int_0^{2\pi} \log \frac{1}{[f(re^{i\theta}), a]} \frac{d\theta}{2\pi}.$$

With these notations the first main theorem of Nevanlinna theory takes the following form.

THEOREM 1.1. *Let f be a meromorphic function in the complex plane, and let $a \in \overline{\mathbb{C}}$. Then*

$$T(r, f) = N(r, a, f) + m(r, a, f) - m(1, a, f)$$

for all $r > 1$.

PROOF. Let

$$T^\circ(r, f) = \frac{1}{\pi} \int_0^r A(t, f) \frac{dt}{t}$$

be the classical Ahlfors-Shimizu characteristic function, and let

$$N^\circ(r, a, f) = \int_0^r (n(t, a, f) - n(0, a, f) \frac{dt}{t} + n(0, a, f) \log r$$

[2]Hence, $[0, \infty] = 1$. This is how it should be, since in our case the radius of the Riemann sphere is $1/2$.

be the corresponding counting function. Then

$$T(r,f) = T^\circ(r,f) - T^\circ(1,f)$$

and

$$N(r,a,f) = N^\circ(r,a,f) - N^\circ(1,a,f)$$

for all $r > 1$. On the other hand, by the first main theorem (see, e.g., [**2**, Theorems 1.3.1 and 1.11.3]) it follows that there exists a constant $c_{fmt}(f,a)$, depending only on f and a, such that

$$T^\circ(r,f) = m(r,a,f) + N^\circ(r,a,f) + c_{fmt}(f,a)$$

for all $r > 1$. Therefore

$$\begin{aligned} T(r,f) &= T^\circ(r,f) - T^\circ(1,f) \\ &= N^\circ(r,a,f) - N^\circ(1,a,f) + m(r,a,f) - m(1,a,f) \\ &= N(r,a,f) + m(r,a,f) - m(1,a,f) \end{aligned}$$

for all $r > 1$. □

Theorem 1.1 implies that the *Nevanlinna defect*

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}$$

can be also written as

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}.$$

We also define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

Finally, we introduce an important modification of the proximity function due to Yamanoi.

Definition 1.2 (Yamanoi). Let f be a transcendental meromorphic function in the complex plane. Then

$$\overline{m}_{d,q}(r,f) = \sup_{(a_1,\dots,a_q)\in(\mathcal{R}_d)^q} \int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi},$$

where $\mathcal{R}_d$ be the set of all rational functions of degree less than or equal to d including the constant function which is identically equal to ∞.

1.2. Yamanoi's theorem and Mues' conjecture

Hayman [**5**] observed that

$$\sum_{a\in\mathbb{C}} \delta(a,f') \le \frac{3}{2} \tag{1.1}$$

using the fact that the derivative function f' has only multiple poles. This follows immediately by the following theorem due to Hayman [**5**].

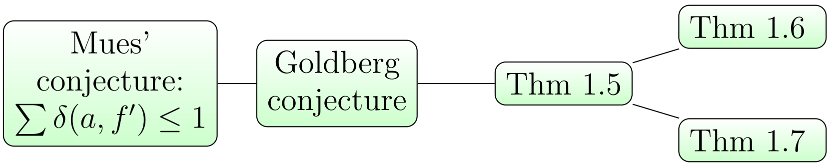

FIGURE 1.1. Yamanoi's Theorem 1.5 implies Goldberg and Mues' conjectures.

THEOREM 1.3. *Let f be a non-constant function meromorphic in the complex plane. Then*

$$\sum_{a \in \mathbb{C}} \Theta(a, f') \leq \frac{3}{2}. \tag{1.2}$$

In particular f' assumes every finite value with at most one exception infinitely often.

PROOF. If $f(z)$ has a pole of order p, then f' has a pole of order $p + 1 \geq 2$. Therefore,

$$\overline{N}(r, f') \leq \frac{1}{2} N(r, f') \leq \frac{1}{2} T(r, f'),$$

and so

$$\Theta(\infty, f') = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, f')}{T(r, f')} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore, by the Nevanlinna defect relation,

$$\sum_{a \in \mathbb{C}} \Theta(a, f') \leq 2 - \Theta(\infty, f') \leq \frac{3}{2}.$$

□

Ishizaki [**9**] and Yang [**20**] showed that (1.1) holds with the constant 3/2 replaced by 4/3 (see also [**15**]). Mues [**10**] conjectured that the best possible upper bound is, in fact, 1.

THEOREM 1.4 (Mues' conjecture). *Let f be a meromorphic function in the complex plane such that the derivative function f' is non-constant. Then*

$$\sum_{a \in \mathbb{C}} \delta(a, f') \leq 1. \tag{1.3}$$

Mues' conjecture was shown to be true by Yamanoi [**19**], who proved the following extension of the Gol'dberg conjecture.

THEOREM 1.5 (Yamanoi). *Let f be a transcendental meromorphic function in the complex plane, let $k \geq 2$ be an integer, and let $\varepsilon > 0$. If A is a finite set of complex numbers, then*

$$(k - 1)\overline{N}(r, \infty, f) + \sum_{a \in A} N_1(r, a, f) \leq N(r, 0, f^{(k)}) + \varepsilon T(r, f)$$

for all $r > e$ outside a set $E \subset (e, \infty)$ of logarithmic density 0*, where E depends only on f, k, ε and A.*

Theorem 1.5, in the special case where $k = 2$ and $A = \emptyset$, implies the Gol'dberg conjecture. We will now show how Theorem 1.4 follows from Theorem 1.5.

Proof of Theorem 1.4 using Theorem 1.5. Suppose first that f is rational. Then, since $\delta(a, g) = 0$ for any rational function g and for all $a \in \overline{\mathbb{C}}$ such that $a \neq g(\infty)$, it follows, in particular, that $\delta(a, f') = 0$ for all $a \neq f'(\infty)$. Therefore (1.3) holds in this case.

Suppose now that f is transcendental, and let $a_1, a_2, \ldots, a_q \in \mathbb{C}$ be distinct. By the second main theorem, it follows that, for any transcendental meromorphic function g,

$$\begin{aligned}\sum_{1 \leq i \leq q} m(r, a_i, g) &\leq 2T(r, g) - m(r, \infty, g) - N(r, 0, g') + N(r, \infty, g') \\ &\quad - 2N(r, \infty, g) + o(T(r, g)) \\ &= T(r, g) - N(r, 0, g') + \overline{N}(r, \infty, g) + o(T(r, g)),\end{aligned}$$

where $r \to \infty$ outside of an exceptional set E of finite linear measure. Hence, in particular, we have

$$\sum_{1 \leq i \leq q} m(r, a_i, f') \leq T(r, f') - N(r, 0, f'') + \overline{N}(r, \infty, f') + o(T(r, f')) \tag{1.4}$$

as $r \to \infty$, $r \notin E$. Let $\varepsilon > 0$ be arbitrary. Then, by Theorem 1.5 with $k = 2$ and $A = \emptyset$, it follows that

$$\overline{N}(r, \infty, f) - N(r, 0, f'') = \overline{N}(r, \infty, f') - N(r, 0, f'') \leq \varepsilon T(r, f) \tag{1.5}$$

for all $r > e$ outside of an exceptional set E' of logarithmic density zero. Now, by combining inequalities (1.4) and (1.5), it follows that

$$\sum_{1 \leq i \leq q} m(r, a_i, f') \leq T(r, f') + \varepsilon T(r, f) + o(T(r, f')) \tag{1.6}$$

as $r \to \infty$ outside of the set $E \cup E'$.

According to a theorem due to Hayman and Miles [**6**], there exists a set $E'' \subset (e, \infty)$ satisfying

$$\overline{\log \operatorname{dens}}\, E'' = \limsup_{r \to \infty} \frac{1}{\log r} \int_{[e, r] \cap E''} \frac{dt}{t} < 1$$

and such that

$$\limsup_{\substack{r \to \infty \\ r \notin E''}} \frac{T(r, f)}{T(r, f')} < 3e + 1.$$

Therefore, by (1.6), and using the fact that

$$\overline{\log \operatorname{dens}}\, (E \cup E' \cup E'') < 1,$$

it follows that

$$\sum_{1\le i\le q}\delta(a_i,f')\le \limsup_{\substack{r\to\infty\\ r\notin E\cup E'\cup E''}}\frac{\sum_{1\le i\le q}m(r,a_i,f')}{T(r,f')}\le 1+\varepsilon(3e+1).$$

Since $\varepsilon>0$ is arbitrary, we can allow $\varepsilon\to 0$ to obtain

$$\sum_{1\le i\le q}\delta(a_i,f')\le 1,$$

which proves Theorem 1.4. □

Following [**19**], we will next introduce upper and lower estimates for the Yamanoi's proximity function $\overline{m}$, and show that they can be combined to prove Theorem 1.5.

The following theorem implies a lower estimate for $\overline{m}$.

THEOREM 1.6. *Let f be a transcendental meromorphic function in the complex plane, let $k\in\mathbb{N}$ and let $\varepsilon>0$. If $\nu:(e,\infty)\to\mathbb{N}$ is a function such that*

$$\nu(r)\sim\left(\log^+\frac{T(r,f)}{\log r}\right)^{20}, \tag{1.7}$$

then

$$\begin{aligned}2T(r,f)+(k-1)\overline{N}(r,\infty,f)\le{}&\overline{m}_{k-1,\nu(r)}(r,f)+N(r,0,f^{(k)})\\&+N_1(r,\infty,f)+\varepsilon T(r,f)\end{aligned}$$

for all $r>e$ outside an exceptional set of logarithmic density zero.

The following theorem gives an upper estimate for $\overline{m}$.

THEOREM 1.7. *Let f be a transcendental meromorphic function in the complex plane, let $d,q\in\mathbb{N}$ and let $\varepsilon>0$. If $B\subset\overline{\mathbb{C}}$ is a finite set of points in the Riemann sphere, and if $p=\#B$, then*

$$\overline{m}_{d,q}(r,f)+\sum_{a\in B}N_1(r,a,f)\le(2+\varepsilon)T(r,f)+\frac{(p+q)^{17}}{\varepsilon^4}T(r,f)^{4/5}(\log r)^{1/5}$$

for all $r>0$ outside a set of finite linear measure $E_{f,d}$ only depending on f and d.

We will now show that Theorem 1.5 follows by combining the lower and upper estimates for $\overline{m}$ of Theorems 1.6 and 1.7, respectively.

PROOF OF THEOREM 1.5. Let A be a finite set of complex numbers, let $\varepsilon>0$ and let $\nu:(e,\infty)\to\mathbb{N}$ be a function satisfying (1.7). By applying Theorem 1.7 with $B=A\cup\{\infty\}$, $d=k-1$ and $q=\nu(r)$, it follows that

$$\begin{aligned}&\overline{m}_{k-1,\nu(r)}(r,f)+N_1(r,\infty,f)+\sum_{a\in A}N_1(r,a,f)\\&\quad\le(2+\varepsilon)T(r,f)+\frac{(p+\nu(r))^{17}}{\varepsilon^4}T(r,f)^{4/5}(\log r)^{1/5}\end{aligned}\tag{1.8}$$

for all $r > 0$ outside a set of finite linear measure $E_{f,k-1}$. On the other hand, by Theorem 1.6, we have

$$\begin{aligned} & 2T(r,f) + (k-1)\overline{N}(r,\infty,f) - N(r,0,f^{(k)}) + \sum_{a\in A} N_1(r,a,f) \\ & \le \overline{m}_{k-1,\nu(r)}(r,f) + N_1(r,\infty,f) + \sum_{a\in A} N_1(r,a,f) + \varepsilon T(r,f) \end{aligned} \tag{1.9}$$

for all $r > e$ outside an exceptional set of logarithmic density zero. By combining (1.8) and (1.9), it follows that

$$\begin{aligned} & (k-1)\overline{N}(r,\infty,f) + \sum_{a\in A} N_1(r,a,f) \\ & \le N(r,0,f^{(k)}) + 2\varepsilon T(r,f) + \frac{(p+\nu(r))^{17}}{\varepsilon^4} T(r,f)^{4/5} (\log r)^{1/5} \end{aligned} \tag{1.10}$$

for all $r > e$ outside an exceptional set of logarithmic density zero. Since f is transcendental, it follows that

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty. \tag{1.11}$$

Therefore, and using (1.7) and the fact that $p = \#A + 1$ is finite, it follows that

$$\lim_{r\to\infty} \frac{(p+\nu(r))^{17} T(r,f)^{4/5} (\log r)^{1/5}}{T(r,f)} = 0.$$

This, together with (1.10), proves the theorem. □

CHAPTER 2

Growth results on real functions

This chapter consists of a number of growth lemmas of Borel type and other similar results on the growth of real functions.

Some terminology. Let $E, F \subset \mathbb{R}$. If

$$\int_E \frac{dt}{t} < +\infty, \qquad \int_F dt < +\infty$$

then E is of *finite logarithmic measure*, and F is of *finite linear measure.* In every case,

$$\overline{\log\text{dens}}(E) = \limsup_{r\to\infty} \frac{1}{\log r} \int_{[e,r]\cap E} \frac{dt}{t},$$
$$\underline{\log\text{dens}}(E) = \liminf_{r\to\infty} \frac{1}{\log r} \int_{[e,r]\cap E} \frac{dt}{t},$$

are well-defined, and are clalled the *upper logarithmic density* and *lower logarithmic density* of E. If these quantities are equal,

$$\log\text{dens}(E) = \lim_{r\to\infty} \frac{1}{\log r} \int_{[e,r]\cap E} \frac{dt}{t}$$

is the *logarithmic density* of E.

The original Borel lemma [**12**, p. 245] will be used in the proof of Theorem 3.1. Lemma 2.1 says that inequality 2.1 holds for all $r \geq r_0$, where $r \notin E$ such that $\int_E dr < +\infty$.

Lemma 2.1 (Borel). *Let $T : [r_0, +\infty[\to [1, +\infty[$ be continuous and non-decreasing. Then*

$$T\left(r + \frac{1}{T(r)}\right) < 2T(r) \tag{2.1}$$

holds outside a set of finite linear measure.

The error term $\varepsilon T(r, f)$ in Theorem 1.5 may be replaced by $o(T(r, f))$ by applying Lemma 2.2 with the function

$$S(r) = (k-1)\overline{N}(r, \infty, f) + \sum_{a\in A} N_1(r, a, f) - N(r, 0, f^{(k)}).$$

Lemma 2.2. *Let T be a non-decreasing real function, and let $S : (r, \infty) \to (0, \infty)$ be a function such that logarithmic density of the set*

$$E_\varepsilon = \{r > e : S(r) > \varepsilon T(r)\}$$

is zero for all $\varepsilon > 0$. *Then* $S(r) = o(T(r))$ *as* $r \to \infty$ *outside of an exceptional set of logarithmic density zero.*

The following results are used in the proof of Proposition 4.1.

LEMMA 2.3. *Let* $g(r)$ *be a continuous, non-decreasing function in* $[r_0, \infty)$ *with* $g(r_0) \geq 2$, *where* $r_0 > 1$. *Suppose that*

$$\lim_{r\to\infty} g(r) = \infty.$$

Given a fixed positive constant $s > 1$, *we put*

$$\varphi(r) = \frac{1}{(\log(g(r))^s}.$$

Set

$$E = \left\{r \geq r_0; \quad g\left(e^{3\varphi(r)}r\right) \geq 2g(r)\right\}.$$

Then we have

$$\int_{E\cap[r_0,\infty]} \frac{dt}{t} < \infty.$$

By choosing

$$g(r) = \exp\left(\frac{\alpha}{\log\frac{1}{r}}\right)$$

with a suitable $1 \leq \alpha < \infty$, it is seen that Lemma 2.3 fails for $s = 1$.

Namely, in this case

$$\varphi(r) = \frac{1}{\log g(r)} = \frac{\log\frac{1}{r}}{\alpha} = \log\frac{1}{r^\alpha}.$$

We obtain

$$g(e^{\beta\varphi(r)}r) = g\left(\frac{r}{r^{\beta/\alpha}}\right) = g(r^{1-\varepsilon}) = \exp\left(\frac{\alpha}{(\log(1/r))(1-\varepsilon)}\right) \geq 2\exp\left(\frac{\alpha}{\log\frac{1}{r}}\right),$$

for sufficiently large r, when $\alpha \geq 1$.

PROOF. Suppose that E is bounded, then our lemma is trivial. Thus in the following, we assume that E is unbounded.

We define a sequence of positive numbers $r_1, r_2, \ldots$ by the following inductive rule:

$$r_1 = \inf E,$$
$$r_{i+1} = \inf\left(E \cap \left[e^{3\varphi(r_i)}r_i, \infty\right)\right).$$

Since E is closed, we have r_i in E. Hence we have

$$g(r_{i+1}) \geq g\left(e^{\varphi(r_i)}r_i\right) \geq 2g(r_i).$$

Thus, we obtain

$$g(r_n) \geq 2^n. \tag{2.2}$$

This shows that $\lim r_n = \infty$. By the construction of the sequence (r_n), we have

$$E \subset \bigcup_{n=1}^{\infty} \left[r_n, e^{3\varphi(r_n)} r_n\right].$$

Using (2.2), we obtain

$$\int_{r_n}^{e^{3\varphi(r_n)} r_n} \frac{dt}{t} = 3\varphi(r_n) = \frac{3}{(\log g(r_n))^s} \leq \frac{3}{(n \log 2)^s}.$$

Thus, we conclude

$$\int_{E \cap [r_0, \infty]} \frac{dt}{t} \leq \sum_{n=1}^{\infty} \int_{r_n}^{e^{3\varphi(r_n)} r_n} \frac{dt}{t} \leq \frac{3}{(\log 2)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty.$$

This proves our lemma. □

COROLLARY 2.4. *Let f be a transcendental meromorphic function in the complex plane and set*

$$\lambda(r) = \min \left\{ 1, \left(\log^+ \frac{T(r,f)}{\log r} \right)^{-1} \right\}$$

We have

$$T(e^{3\lambda(r)^2} r) \leq 3T(r)$$

for all $r > e$ outside a set $E \subset [e, \infty)$ of finite logarithmic measure $\int_E (dt/t) < \infty$.

PROOF. For $r > 1$, the function $T(r)/\log r$ is a continuous, non-decreasing function. Since f is transcendental, we have $\lim_{r\to\infty} T(r)/\log r = \infty$. We apply Lemma 2.3 to obtain (set $\varphi = \lambda^2$)

$$\frac{T(e^{3\lambda(r)^2} r)}{\log(e^{3\lambda(r)^2} r)} < 2 \frac{T(r)}{\log r}$$

for all $r > e$ outside a set of finite logarithmic measure. Here

$$\frac{\log \left(e^{3\lambda(r)^2} r\right)}{\log r} = \frac{\log r + 3\lambda(r)^2}{\log r} = 1 + \frac{3\lambda(r)^2}{\log r}.$$

Hence we obtain

$$T(e^{3\lambda(r)^2} r) < 2 \left(1 + \frac{3\lambda(r)^2}{\log r} \right) T(r)$$

for all $r > e$ outside a set of finite logarithmic measure. Since

$$\lim_{r\to\infty} \frac{\lambda(r)^2}{\log r} = \lim_{r\to\infty} \min \left\{ \frac{1}{\log r}, \left((\log r) \left(\log^+ \frac{T(r,f)}{\log r} \right)^2 \right)^{-1} \right\} = 0$$

we complete the proof of our corollary. □

LEMMA 2.5. *Let $F \subset \mathbb{R}_{>e}$ be a measurable set. Let $\varphi : [e,\infty) \to (0,\infty)$ be a positive continuous and non-increasing function. Assume that the set*

$$E_\varepsilon = \left\{ r \geq e; \quad \int_{F \cap [r, e^{\varphi(r)} r]} \frac{dt}{t} > \varepsilon\varphi(r) \right\}$$

has finite logarithmic measure for every $\varepsilon > 0$. Then the logarithmic density of F is zero.

These Borel type lemmas can be generalized in many ways. One example with auxiliary functions is [**2**, Lemma 3.3.1].

LEMMA 2.6. *Let*

$$\begin{aligned} F &: [r_0,\infty) \to [e,\infty), \\ \phi &: [r_0,\infty) \to (0,\infty), \\ \xi &: [e,\infty) \to (0,\infty) \end{aligned} \tag{2.3}$$

be non-decreasing and continuous. Let $C > 1$ be a constant and set

$$E = \left\{ r \in [r_0,\infty)\,;\ F\left(r + \frac{\phi(r)}{\xi(F(r))} \right) \geq CF(r) \right\}. \tag{2.4}$$

Then

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)} + \frac{1}{\log C} \int_e^{F(R)} \frac{dx}{x\xi(x)}.$$

EXAMPLE 2.7. Choose $\phi \equiv 1$, $\xi(r) = r$, $F = T$ in Lemma 2.6. Then we see that

$$T\left(r + \frac{1}{T(r)} \right) < CT(r), \quad r \in [r_0,\infty) \setminus E,$$

where

$$\int_{E \cap [r_0, R]} \leq \frac{1}{e} + \frac{1}{\log C} \int_e^{T(R)} \frac{dx}{x^2} = \frac{1}{e} + \frac{1}{\log C} \left(\frac{1}{T(R)} - \frac{1}{e} \right).$$

In February 2019, we discussed the previous Borel-type lemmas. In Yamanoi's manuscript *"On a reversal of the second main theorem for meromorphic functions of finite order"*, there are two other Borel-type lemmas:

LEMMA 2.8. *Let g be a continuous, non-decreasing function in $[e,\infty)$ and $g(e) > 0$. Suppose that*

$$M = \overline{\lim_{r\to\infty}} \frac{\log g(r)}{\log r} < \infty. \tag{2.5}$$

Given $0 < \varepsilon < 1$, put

$$C(\varepsilon) = 2 \cdot 8^{2M/\varepsilon}$$
$$E_\varepsilon = \{r \in [e,\infty)\,;\, g(8r) \geq C(\varepsilon) g(r)\}\,.$$

Then we have

$$\overline{\log \operatorname{dens}}\, E_\varepsilon < \varepsilon.$$

It is clear, that if g grows faster (the condition (2.5) is relaxed), then C cannot be a constant.
• if (2.5), then $g \lesssim r^M$, $C(\varepsilon) = 2 \cdot 8^{2M/\varepsilon}$;
• if

$$g(r) = e^{(\log r)^\alpha},$$

then

$$g(8r) = e^{(\log r + \log 8)^\alpha} \sim g(r)$$

and perhaps C can be taken a constant.
• perhaps

$$M(r) = \frac{\log g(r)}{\log r}, \quad C(\varepsilon, r) = 2 \cdot 8^{2M(r)/\varepsilon}?$$

• $g(r) = e^r$, $C(\varepsilon) \geq e^{7r}$.

LEMMA 2.9. *Let $F \subset \mathbb{R}_{>e}$ be a measurable set, and let $\alpha \geq 0$. We define a set E by*

$$E = \left\{ r\,;\, \int_{F\,\cap\,[r,2r]} \frac{dt}{t} > \alpha \right\}.$$

Then we have

$$\overline{\log \operatorname{dens}}\, F \leq \frac{\alpha}{\log 2} + \overline{\log \operatorname{dens}}\, E.$$

Proof of Lemma 2.8 If E_ε is bounded, then our lemma is trivial. Thus in the following, we assume that E_ε is not bounded.

We define a sequence of positive numbers $r_1, r_2, \ldots$ by the following inductive rule:

$$r_1 = \inf E_\varepsilon,$$
$$r_{i+i} = \inf(E_\varepsilon \cap [8r_i, \infty)).$$

Since E_ε is a closed set, we have $r_i \in E_\varepsilon$. Hence, we have

$$g(r_{i+1}) \geq g(8r_i) \geq C(\varepsilon)g(r_i). \tag{2.6}$$

Now given large R with $E_\varepsilon \cap [e, R] \neq \emptyset$, there is a positive integer $n(R)$ such that

$$E_\varepsilon \cap [e, R] \subset \bigcup_{i=1}^{n(R)} [r_i, 8r_i]$$

and

$$r_{n(R)} \leq R.$$

Then since

$$\int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} \leq \sum_{i=1}^{n(R)} \int_{r_i}^{8r_i} \frac{dt}{t} \leq n(R) \log 8,$$

we have

$$n(R) \geq \frac{1}{\log 8} \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t}$$

and

$$n(R) \log C(\varepsilon) \geq \frac{\log 2 + \frac{2M}{\varepsilon} \log 8}{\log 8} = \left(\frac{1}{3} + \frac{2M}{\varepsilon}\right) \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t}.$$

Hence by 2.6, we have

$$\begin{aligned}
\log g(R) &\geq \log g(r_{n(R)}) \\
&\geq \log(C(\varepsilon)^{n(R)-1} g(r_1)) \\
&= n(R) \log C(\varepsilon) - \log C(\varepsilon) + \log g(r_1) \\
&\geq \left(\frac{1}{3} + \frac{2M}{\varepsilon}\right) \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} - \log C(\varepsilon) + \log g(r_1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\overline{\lim_{R\to\infty}} \frac{1}{\log R} \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} \\
&\qquad \leq \left(\frac{3\varepsilon}{6M + \varepsilon}\right) \overline{\lim_{R\to\infty}} \frac{\log g(R) + \log C(\varepsilon) - \log g(r_1)}{\log R} < \varepsilon.
\end{aligned}$$

This proves our lemma. □

Proof of Lemma 2.9. Put $G = [e, \infty) \setminus E$. Then G is a closed set. Suppose that G is bounded. In this case, the upper logarithmic density of E is equal to 1, so our lemma is trivial. Hence, in the following, we assume that G is unbounded.

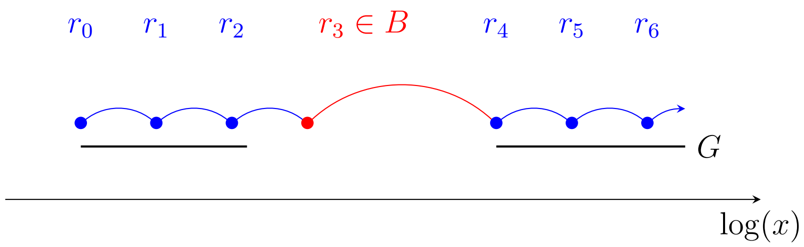

FIGURE 2.1. Sequence $\{r_n\}$.

We define a sequence of positive numbers $\{r_n\}$ by the following inductive rule:

$$r_0 = e$$
$$r_{i+1} = \begin{cases} 2r_i, & r_i \in G \\ \inf[r_i, \infty) \cap G, & r_i \notin G. \end{cases}$$

Since we are assuming that G is unbounded, this sequence is infinite. We observe that

$$r_{i+2} \geq 2r_i. \tag{2.7}$$

Indeed, this is obvious if $r_i \in G$. Suppose that $r_i \notin G$. Then since G is closed, we conclude $r_{i+1} \in G$. Hence $r_{i+2} = 2r_{i+1}$, and we conclude (2.7) for $r_i \notin G$. From (2.7), we see that the sequence $\{r_n\}$ tends to infinity.

Now given $R > e$, there is a non-negative integer $n(R)$ such that

$$r_{n(R)} \leq R < r_{n(R)+1}.$$

We put

$$A = \{i \in \mathbb{Z}_{\geq 0}\,;\, r_i \in G \quad \text{and} \quad i \leq n(R) - 1\},$$
$$B = \{i \in \mathbb{Z}_{\geq 0}\,;\, r_i \notin G \quad \text{and} \quad i \leq n(R) - 1\},$$

so that $\mathbb{Z} \cap [0, n(R) - 1] = A \cup B$.

Claim. For the cardinality of A, we have

$$|A| \leq \frac{\log(R/e)}{\log 2}.$$

Proof of the Claim. We consider the case, where $|A|$ is large, compared to a fixed (small) R. We will have $|B| = 0$. (In general, $|B| \leq |A|$. This is because $r_i \notin G$ implies $r_{i+1} \in G$, and hence $i \in B$ implies $i + 1 \in A$.)

First, let A contain all the integers i and $B = \emptyset$. By the definition of the numbers r_i, we have $r_i = 2^i e$ for all i.

Second, take $R = 2^k e$. Now, R optimally small in that sense that just barely $r_k = R \in [0, R]$.

Now $|A| = |\{0,\dots,n(R)-1\}| = |\{0,\dots,k-1\}| = k$. We obtain

$$2^k = \frac{R}{e}, \quad \text{implying} \quad k\log 2 = \log\frac{R}{e}$$

which implies

$$|A| = k = \frac{\log(R/e)}{\log 2}.$$

□

We have

$$G \subset \bigcup_{i\in A}[r_i, r_{i+1}].$$

In other words G lies below the blue curve in Figure 2.1. Therefore,

$$\bigcup_{i\in B}[r_i, r_{i+1}] \subset E.$$

Hence, we have

$$\begin{aligned}
\int_{[e,\,R]\,\cap\,F}\frac{dt}{t} &= \sum_{i=0}^{n(R)-1}\int_{[r_i,\,r_{i+1}]\,\cap\,F}\frac{dt}{t} + \int_{[r_{n(R)},\,R]\,\cap\,F}\frac{dt}{t}\\
&= \sum_{i\in A}\int_{[r_i,\,r_{i+1}]\,\cap\,F}\frac{dt}{t} + \sum_{i\in B}\int_{[r_i,\,r_{i+1}]\,\cap\,F}\frac{dt}{t} + \int_{[r_{n(R)},\,R]\,\cap\,F}\frac{dt}{t}\\
&\le \alpha(|A|+1) + \int_{[e,\,R]\,\cap\,E}\frac{dt}{t}\\
&\le \alpha\left(\frac{\log(R/e)}{\log 2}+1\right) + \int_{[e,\,R]\,\cap\,E}\frac{dt}{t}\\
&\le \alpha\left(\frac{\log R}{\log 2}+1\right) + \int_{[e,\,R]\,\cap\,E}\frac{dt}{t}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\overline{\lim_{R\to\infty}}\frac{1}{\log R}\int_{[e,\,R]\,\cap\,F}\frac{dt}{t} &\le \alpha\,\overline{\lim_{R\to\infty}}\left(\frac{1}{\log 2}+\frac{1}{\log R}\right) + \overline{\lim_{R\to\infty}}\frac{1}{\log R}\int_{[e,\,R]\,\cap\,E}\frac{dt}{t}\\
&\le \frac{\alpha}{\log 2} + \overline{\log\operatorname{dens}}\,E.
\end{aligned}$$

This proves our lemma. □

CHAPTER 3

Estimates in Nevanlinna Theory

This chapter contains a number of basic estimates needed in the proof of Theorem 1.5, and results related to it.

3.1. Uniform second main theorem

The next result due to Yamanoi [**19**] is an inverse inequality to the Nevanlinna's second main theorem with Yamanoi's proximity function $\overline{m}$.

THEOREM 3.1 (Uniform second main theorem). *Let f be a transcendental meromorphic function on the complex plane, let $q \geq 2$ be an integer, and let $a_1, \ldots, a_q \in \overline{\mathbb{C}}$ be distinct. Then*

(3.1)
$$\int_0^{2\pi} \max_{1\leq j\leq q} \log \frac{1}{[f(re^{i\theta}), a_j]} \frac{d\theta}{2\pi} + \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) \leq 2T(r,f) + 3\log T(r,f) + 2\log q$$

for all $r > 1$ outside an exceptional set E of finite linear measure, which depends only on f.

The q-dependence of right-hand side of (3.1) is explicit - this is clearly a good thing.

We first need the following lemma[1] in order to prove the uniform second main theorem. Define the spherical derivative as $f^{\#} = |f'|/(1+|f|^2)$.

LEMMA 3.2. *Let f be a transcendental meromorphic function on the complex plane. Then*

$$-\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} = 2T(r,f) - \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) - \int_0^{2\pi} \log f^{\#}(e^{i\theta}) \frac{d\theta}{2\pi}$$

for all $r > 1$.

PROOF. Since

$$N(r,0,f') = \sum_{a\in\mathbb{C}} N_1(r,a,f)$$

[1]This lemma is used in Yamanoi's "Reversal paper" and can be found from [**2**, Proposition 2.4.2].

for all $r > 1$, it follows by the Jensen formula (see, e.g., [**2**, Corollary 1.2.1]) that

$$\begin{aligned}
-\int_0^{2\pi} \log f^{\#}(re^{i\theta})\frac{d\theta}{2\pi} &= -\int_0^{2\pi} \log|f'(re^{i\theta})|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log\left(1+|f(re^{i\theta})|^2\right)\frac{d\theta}{2\pi} \\
&= N(r,\infty,f') - N(r,0,f') + \int_0^{2\pi} \log\left(1+|f(re^{i\theta})|^2\right)\frac{d\theta}{2\pi} \\
&\quad - \int_0^{2\pi} \log|f'(e^{i\theta})|\frac{d\theta}{2\pi} \\
&= N(r,\infty,f) + \overline{N}(r,\infty,f) - \sum_{a\in\mathbb{C}} N_1(r,a,f) \\
&\quad + 2m(r,\infty,f) - \int_0^{2\pi} \log|f'(e^{i\theta})|\frac{d\theta}{2\pi}
\end{aligned}$$

for all $r > 1$. Therefore, Theorem 1.1 yields

$$\begin{aligned}
-\int_0^{2\pi} \log f^{\#}(re^{i\theta})\frac{d\theta}{2\pi} &= N(r,\infty,f) + \overline{N}(r,\infty,f) - \sum_{a\in\mathbb{C}} N_1(r,a,f) \\
&\quad + 2T(r,f) - 2N(r,\infty,f) + 2m(1,\infty,f) \\
&\quad - \int_0^{2\pi} \log|f'(e^{i\theta})|\frac{d\theta}{2\pi} \\
&= 2T(r,f) - \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) - \int_0^{2\pi} \log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi}
\end{aligned}$$

for all $r > 1$. □

Proof of Theorem 3.1. Letting $1 < r < \rho$ and $0 < \alpha < 1$, it follows by the Gol'dberg-Grinshtein estimate (see [**2**, Theorem 3.2.2]) that

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{\alpha} \frac{d\theta}{2\pi} \le C(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^{\alpha} (2T(\rho,f)+2C_{f,0})^{\alpha},$$

where $C(\alpha) = 2^{\alpha} + (8+2^{\alpha+1})\sec(\alpha\pi/2)$ and $C_{f,0} = \sup_{a\in\overline{\mathbb{C}}} m(1,f,a)$. By using the the estimate

$$\frac{f^{\#}}{[f,0]} \le \left|\frac{f'}{f}\right|,$$

it follows that

$$\int_0^{2\pi} \left(\frac{f^{\#}(re^{i\theta})}{[f(re^{i\theta}),0]}\right)^{\alpha} \frac{d\theta}{2\pi} \le C(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^{\alpha} (2T(\rho,f)+2C_{f,0})^{\alpha}$$

for all $1 < r < \rho$ and $0 < \alpha < 1$. Hence, for $a \in \overline{\mathbb{C}}$, using a rotation of Riemann sphere

$$z \mapsto \frac{1 - \frac{1}{a}z}{z + \frac{1}{\bar{a}}}, \quad a \neq \infty,$$
$$z \mapsto \frac{1}{z}, \qquad a = \infty,$$

which takes a to 0 and with respect to which the spherical derivative is invariant (see, e.g., [**4**, p. 19]), we obtain

$$\int_0^{2\pi} \left(\frac{f^{\#}(re^{i\theta})}{[f(re^{i\theta}), a]}\right)^{\alpha} \frac{d\theta}{2\pi} \leq C(\alpha) \left(\frac{\rho}{r(\rho - r)}\right)^{\alpha} (2T(\rho, f) + 2C_{f,0})^{\alpha}.$$

Thus using the concavity of log, and

$$(x+y)^{\alpha} \leq x^{\alpha} + y^{\alpha}, \quad x, y \in (0, \infty), \quad 0 < \alpha < 1,$$

we have

$$\begin{aligned}
&\int_0^{2\pi} \max_{1 \leq i \leq q} \log \frac{1}{[f(re^{i\theta}), a_i]} \frac{d\theta}{2\pi} \\
&\leq \int_0^{2\pi} \log \sum_{1 \leq i \leq q} \frac{1}{[f(re^{i\theta}), a_i]} \frac{d\theta}{2\pi} \\
&= -\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} + \frac{1}{\alpha} \int_0^{2\pi} \log \left(\sum_{1 \leq i \leq q} \frac{f^{\#}(re^{i\theta})}{[f(re^{i\theta}), a_i]}\right)^{\alpha} \frac{d\theta}{2\pi} \\
&\leq -\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} + \frac{1}{\alpha} \int_0^{2\pi} \log \sum_{1 \leq i \leq q} \left(\frac{f^{\#}(re^{i\theta})}{[f(re^{i\theta}), a_i]}\right)^{\alpha} \frac{d\theta}{2\pi} \\
&\leq -\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} + \frac{1}{\alpha} \log \sum_{1 \leq i \leq q} \int_0^{2\pi} \left(\frac{f^{\#}(re^{i\theta})}{[f(re^{i\theta}), a_i]}\right)^{\alpha} \frac{d\theta}{2\pi} \\
&\leq -\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} + \log(T(\rho, f) + C_{f,0}) + \log \frac{\rho}{r(\rho - r)} \\
&\quad + \frac{1}{\alpha} \log C(\alpha) + \frac{1}{\alpha} \log q + \log 2.
\end{aligned}$$

Since by Lemma 3.2

$$-\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} = 2T(r, f) - \sum_{a \in \overline{\mathbb{C}}} N_1(r, a, f) - \int_0^{2\pi} \log f^{\#}(e^{i\theta}) \frac{d\theta}{2\pi},$$

we conclude

$$\begin{aligned}&\int_0^{2\pi} \max_{1\le i\le q} \log \frac{1}{[f(re^{i\theta}), a_i]} \frac{d\theta}{2\pi} + \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f)\\ &\le 2T(r,f) + \log(T(\rho,f)+C_{f,0}) + \log\frac{\rho}{r(\rho-r)}\\ &\quad + \frac{1}{\alpha}\log C(\alpha) + \frac{1}{\alpha}\log q + \log 2 - \int_0^{2\pi} \log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi}.\end{aligned}$$

Now let $\alpha = \frac{1}{2}$ and $\rho = r + 1/T(r,f)$. We set

$$E = \left\{ r > 1; T\left(r + \frac{1}{T(r,f)}, f\right) > 2T(r,f)\right\}.$$

Then by Borel's growth lemma [**12**, p. 245], the set E is of finite linear measure, which only depends on f. There exists $r_0 > 1$ such that

$$\begin{aligned}&\log(2T(r,f)+C_{f,0}) + \log\left(1 + \frac{1}{rT(r,f)}\right) + \frac{1}{\alpha}\log C(\alpha)\\ &\quad + \log 2 - \int_0^{2\pi}\log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi} < 2\log T(r,f)\end{aligned} \tag{3.2}$$

for all $r > r_0$. Now we obtain (3.1) for all $r > 1$ outside the exceptional set $E \cup [1, r_0]$ of finite linear measure, which only depends on f. □

The next result is an application of Theorem 1.6 and of the uniform Nevanlinna's second main theorem.

THEOREM 3.3. *Let f be a transcendental meromorphic function on the complex plane, and let $\nu : (0,\infty) \to \mathbb{N}$ be a function that satisfies (1.7). Then*

$$\overline{m}_{0,\nu(r)}(r,f) + \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) = 2T(r,f) + o(T(r,f))$$

as $r\to\infty$ outside a set of logarithmic density zero.

PROOF. First note that by definition

$$\overline{m}_{0,q}(r,f) = \sup_{(a_1,\dots,a_q)\in\overline{\mathbb{C}}^q} \int_0^{2\pi} \max_{1\le j\le q}\log\frac{1}{[f(re^{i\theta}), a_j]}\frac{d\theta}{2\pi}.$$

By applying Theorem 1.6 with $k=1$, and Lemma 2.2, we have

$$2T(r,f) \le \overline{m}_{0,\nu(r)}(r,f) + N(r,0,f') + N_1(r,\infty,f) + o(T(r,f)) \tag{3.3}$$

as $r\to\infty$ outside of a set of logarithmic density zero. Since

$$N_1(r,\infty,f) + N(r,0,f') = \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f),$$

inequality (3.3) becomes

$$2T(r,f) \le \overline{m}_{0,\nu(r)}(r,f) + \sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) + o(T(r,f)), \tag{3.4}$$

where $r \to \infty$ outside of a set of logarithmic density zero.

By Theorem 3.1 we have, for any collection of $q \ge 2$ distinct points $a_1, \ldots, a_q \in \overline{\mathbb{C}}$, the inequality (3.1). Therefore, by choosing $q = \nu(r)$ and taking the supremum over all such collections $a_1, \ldots, a_q \in \overline{\mathbb{C}}$, it follows that

$$\overline{m}_{0,\nu(r)}(r,f)+\sum_{a\in\overline{\mathbb{C}}} N_1(r,a,f) \le 2T(r,f)+3\log T(r,f)+2\log \nu(r) \tag{3.5}$$

for all $r > 1$ outside of an exceptional set E of finite linear measure that depends only on f. Since by (1.7), we have

$$3\log T(r,f) + 2\log \nu(r) = o(T(r,f))$$

as $r \to \infty$, the assertion follows by combining (3.4) and (3.5). □

3.2. Basic inequalities

Let $f(z)$ and $a(z)$ be meromorphic functions in the complex plane such that $f \not\equiv a$. We will next extend the definitions of the Nevanlinna functions for f and a. First, the proximity function is defined as

$$m(r,a,f) = \int_0^{2\pi} \log \frac{1}{[f(re^{i\theta}), a(re^{i\theta})]} \frac{d\theta}{2\pi}.$$

In order to define the counting function, let $f = g/h$ be a reduced representation of f, meaning that g and h are entire without common zeros. Similarly, let $a = b/c$ be a reduced representation of a. Then $n(t,a,f)$ is defined by

$$n(t,a,f) = \sum_{z\in D(t)} \operatorname{ord}_z^+(gc - hb),$$

and the counting function as

$$N(r,a,f) = \int_1^r n(t,a,f)\frac{dt}{t}.$$

Similarly, the reduced counting function is defined by

$$\overline{N}(r,a,f) = \int_1^r \overline{n}(t,a,f)\frac{dt}{t},$$

where

$$\overline{n}(t,a,f) = \sum_{z\in D(t)} \min\{1, \operatorname{ord}_z^+(gc - hb)\}.$$

In addition, we define

$$N_1(r,a,f) = N(r,a,f) - \overline{N}(r,a,f).$$

LEMMA 3.4. *Let f and a be meromorphic functions in the complex plane such that $f \not\equiv a$. If $r > \delta > 0$, then*

$$\frac{1}{\pi}\int_\delta^r \left(\int_{D(t)} f^*\omega\right)\frac{dt}{t} + \frac{1}{\pi}\int_\delta^r \left(\int_{D(t)} a^*\omega\right)\frac{dt}{t} = \int_\delta^r n(t,a,f)\frac{dt}{t} + m(r,a,f) - m(\delta,a,f),$$

and

$$T(r,f) + T(r,a) = N(r,a,f) + m(r,a,f) - m(1,a,f).$$

LEMMA 3.5. *Let $a \in \mathcal{R}_d$, and let f be a meromorphic function such that $f \notin \mathcal{R}_d$. Then*

$$m(1,a,f) \le C,$$

where $C > 0$ is a constant depending only on d and f.

PROOF. Suppose, on the contrary to the assertion, that there exists a sequence $a_n(z) \in \mathcal{R}_d$, $n \in \mathbb{N}$, such that

$$m(1,a_n,f) \to \infty \tag{3.6}$$

as $n \to \infty$. By taking a suitable subsequence, if necessary, we may assume that $(a_n(z))_{n\in\mathbb{N}}$ converges locally uniformly to $a(z) \in \mathcal{R}_d$ outside a finite set of points in $\mathbb{C}$. We can therefore choose a constant $\delta \in (0,1)$ such that $(a_n(z))_{n\in\mathbb{N}}$ converges to $a(z)$ uniformly on $\{z \in \mathbb{C} : |z| = \delta\}$, and

$$\min_{0\le\theta\le 2\pi}[f(\delta e^{i\theta}), a(\delta e^{i\theta})] > 0.$$

Therefore,

$$\sup_{n\in\mathbb{N}} m(\delta,a_n,f) < \infty. \tag{3.7}$$

On the other hand, by Lemma 3.4, we have

$$\frac{1}{\pi}\int_\delta^1 \left(\int_{D(t)} f^*\omega\right)\frac{dt}{t} + \frac{1}{\pi}\int_\delta^1 \left(\int_{D(t)} a_n^*\omega\right)\frac{dt}{t} = \int_\delta^1 n(t,a_n,f)\frac{dt}{t} + m(1,a_n,f) - m(\delta,a_n,f), \tag{3.8}$$

for all $n \in \mathbb{N}$. But since $a_n \in \mathcal{R}_d$ for all $n \in \mathbb{N}$, it follows that

$$\int_{D(t)} a_n^*\omega \le d\pi$$

for all $n \in \mathbb{N}$, and so

$$\frac{1}{\pi}\int_\delta^1 \left(\int_{D(t)} a_n^*\omega\right)\frac{dt}{t} \le -d\log\delta$$

for all $n \in \mathbb{N}$. Therefore, taking into account that

$$\int_\delta^1 n(t,a_n,f)\frac{dt}{t} \ge 0,$$

equation (3.8) yields

$$m(\delta, a_n, f) \geq m(1, a_n, f) - \frac{1}{\pi}\int_\delta^1 \left(\int_{D(t)} f^*\omega\right)\frac{dt}{t} + d\log\delta,$$

and so, by (3.6), $m(\delta, a_n, f) \to \infty$ as $n \to \infty$. This contradicts with (3.7), and thus the assertion follows. □

Now we can prove the following lemma, which shows that Yamanoi's proximity function is finite.

LEMMA 3.6. *Let f be a meromorphic function such that $f \notin \mathcal{R}_d$, and let $q \in \mathbb{N}$. Then $\overline{m}_{d,q}(r, f)$ is finite for all $r > 0$.*

PROOF. Let $a \in \mathcal{R}_d$. By Lemma 3.5, it follows that

$$m(r, a, f) = m(1, a(rz), f(rz)) \leq C_{f(rz),d},$$

where

$$C_{f,d} = \sup_{a\in\mathcal{R}_d} m(1, a, f).$$

Therefore, for $(a_1, \ldots, a_q) \in (\mathcal{R}_d)^q$, we have

$$\int_0^{2\pi} \max_{1\leq j\leq q} \log\frac{1}{[f(re^{i\theta}), a(re^{i\theta})]}\frac{d\theta}{2\pi} \leq \sum_{j=1}^q m(r, a_j, f) \leq qC_{f(rz),d},$$

and so, by taking a supremum over $(\mathcal{R}_d)^q$, it follows that

$$\overline{m}_{d,q}(r, f) \leq qC_{f(rz),d}$$

for all $r > 0$. Thus the assertion follows. □

Next we prove the following lemma [**19**, Lemma 2.4].

LEMMA 3.7. *Let f be a meromorphic function with $f \notin \mathcal{R}_d$. For $a_1, a_2, a_3, a_4 \in \mathcal{R}_d - \{\infty\}$ with $a_1a_4 - a_2a_3 \neq 0$, we have*

$$T\left(r, \frac{a_1f - a_2}{a_3f - a_4}\right) \leq T(r, f) + 2C_{f,2d} + 8d\log r.$$

Before proving this lemma, we shall recall the Nevanlinna theory for holomorphic curves $F : \mathbb{C} \to \mathbb{P}^k$ into the projective space [**13**, p. 101]. The case $k = 1$ reduces to the theory of meromorphic functions. Let $[X_1 : \ldots : X_{k+1}]$ be homogeneus coordinate for $\mathbb{P}^k$. Let $F : \mathbb{C} \to \mathbb{P}^k$ be a holomorphic curve with a reduced representation $[g_1 : \ldots : g_{k+1}]$. By definition, $g_1, \ldots, g_{k+1}$ are entire functions with no common zero. We set

$$T(r, F) = \int_1^r \int_{\mathbb{C}(t)} dd^c \log\left(\sum_{i=1}^{k+1} |g_i|^2\right)\frac{dt}{t}. \tag{3.9}$$

Let $H \subset \mathbb{P}^k$ be a hyperplane defined by $\{X_1 = 0\}$. We set

$$N(r, H, F) = N(r, 0, g_1).$$

We define the Weil function $\lambda_H : \mathbb{P}^k \setminus H \to \mathbb{R}$ for H by

$$\lambda_H = \frac{1}{2}\log\left(1+\sum_{i=2}^{k+1}\frac{|X_i|^2}{|X_1|^2}\right). \tag{3.10}$$

We set

$$m(r,H,F) = \frac{1}{2\pi}\int_0^{2\pi}\lambda_H(F(re^{i\theta}))d\theta.$$

Then we have the first main theorem

$$T(r,F) = N(r,H,F) + m(r,H,F) - m(1,H,F). \tag{3.11}$$

Proof of Lemma 3.7. By Lemma 3.4 we have

$$T(r,f) + T(r,a) = N(r,a,f) + m(r,a,f) - m(1,a,f)$$

for distinct meromorphic functions $f(z)$ and $a(z)$. Setting $a(z) \equiv 0$, it follows that

$$T(r,f) = N(r,0,f) + m(r,0,f) - m(1,0,f),$$

for any meromorphic function f. Similarly, by Lemma 3.4 it follows that

$$T\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right) = N\left(r,0,\frac{a_1f-a_2}{a_3f-a_4}\right) + m\left(r,0,\frac{a_1f-a_2}{a_3f-a_4}\right) - m\left(1,0,\frac{a_1f-a_2}{a_3f-a_4}\right).$$

Now

$$\frac{a_1f-a_2}{a_3f-a_4} = 0,$$

only if $f = a_2/a_1$ or $a_3 = \infty$ or $a_4 = \infty$. Therefore

$$N\left(r,0,\frac{a_1f-a_2}{a_3f-a_4}\right) \le N(r,f,a_2/a_1) + 2d\log r$$

and we have

$$T\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right) \le N(r,a_2/a_1,f) + m\left(r,0,\frac{a_1f-a_2}{a_3f-a_4}\right) - m\left(1,0,\frac{a_1f-a_2}{a_3f-a_4}\right) + 2d\log r. \tag{3.12}$$

We next estimate the proximity functions on the right-hand side. Let $a,b,c,d \in \mathbb{C}$ with $ad - bc \neq 0$. Setting

$$\Lambda(a,b,c,d) = \frac{1}{2}\log\left(1+\frac{|c|^2+|d|^2}{|a|^2+|b|^2}\right) = \frac{1}{2}\log\left(\frac{|a|^2+|b|^2+|c|^2+|d|^2}{|a|^2+|b|^2}\right)$$

we have

$$\Lambda(a,b,c,d) - \log\frac{1}{[w,d/c]} \le \log\frac{1}{[(aw-b)/(cw-d),0]} \le \Lambda(a,b,c,d) + \log\frac{1}{[w,b/a]}. \tag{3.13}$$

In order to show that inequality (3.13) holds, we will use the inequalities

$$|cw - d|^2 \le (|c|^2 + |d|^2)(1 + |w|^2);$$
$$|aw - b|^2 \le (|a|^2 + |b|^2)(1 + |w|^2),$$

which hold by Cauchy-Schwarz inequality. We see that

$$\left[w, \frac{d}{c}\right]^2 = \frac{\left|w - \frac{d}{c}\right|^2}{(1 + |w|^2)\left(1 + \left|\frac{d}{c}\right|^2\right)} \frac{|c|^2}{|c|^2} = \frac{|cw - d|^2}{(1 + |w|^2)(|c|^2 + |d|^2)},$$

and hence

$$\begin{aligned} \Lambda(a,b,c,d) - \log \frac{1}{\left[w, \frac{d}{c}\right]} &= \frac{1}{2}\log\left[\left(1 + \frac{|c|^2 + |d|^2}{|a|^2 + |b|^2}\right)\left(\frac{|cw - d|^2}{(1 + |w|^2)(|c|^2 + |d|^2)}\right)\right] \\ &\le \frac{1}{2}\log\left(1 + \frac{|cw - d|^2}{(|a|^2 + |b|^2)(1 + |w|^2)}\right) \\ &\le \frac{1}{2}\log\left(1 + \frac{|cw - d|^2}{|aw - b|^2}\right) \\ &= \log\frac{1}{[0, (aw - b)/(cw - d)]}, \end{aligned}$$

where the last equality follows from

$$\frac{1}{[0, \alpha/\beta]^2} = \frac{1 + |\alpha/\beta|^2}{|\alpha/\beta|^2} = \frac{|\alpha|^2 + |\beta|^2}{|\alpha|^2} = 1 + \frac{|\beta|^2}{|\alpha|^2}.$$

Similarly,

$$\begin{aligned} \Lambda(a,b,c,d) + \log \frac{1}{\left[w, \frac{b}{a}\right]} &= \frac{1}{2}\log\left[\left(1 + \frac{|c|^2 + |d|^2}{|a|^2 + |b|^2}\right)\left(\frac{(1 + |w|^2)(|a|^2 + |b|^2)}{|aw - b|^2}\right)\right] \\ &\ge \frac{1}{2}\log\left(1 + \frac{(|c|^2 + |d|^2)(1 + |w|^2)}{|aw - b|^2}\right) \\ &\ge \frac{1}{2}\log\left(1 + \frac{|cw - d|^2}{|aw - b|^2}\right) \\ &= \log\frac{1}{[0, (aw - b)/(cw - d)]}. \end{aligned}$$

This shows that (3.13) holds.

Now by (3.12) and (3.13), we have

$$\begin{aligned} T\left(r, \frac{a_1 f - a_2}{a_3 f - a_4}\right) \le\ & N(r, a_2/a_1, f) + m(r, a_2/a_1, f) + m(1, a_4/a_3, f) \\ & + \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(re^{i\theta})\frac{d\theta}{2\pi} \\ & - \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(e^{i\theta})\frac{d\theta}{2\pi} + 2d\log r. \end{aligned}$$

By Lemmas 3.4 and 3.5, we have

$$N(r, a_2/a_1, f) + m(r, a_2/a_1, f) \le T(r, f) + C_{f,2d} + 2d \log r.$$

Hence, we obtain

$$\begin{aligned} T\left(r, \frac{a_1 f - a_2}{a_3 f - a_4}\right) &\le T(r, f) + 2C_{f,2d} + \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(re^{i\theta}) \frac{d\theta}{2\pi} \\ &\quad - \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(e^{i\theta}) \frac{d\theta}{2\pi} + 4d \log r. \end{aligned}$$

Finally, we claim

(3.14)
$$\int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(re^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(e^{i\theta}) \frac{d\theta}{2\pi} \le 4d \log r.$$

Indeed we have

$$\Lambda(a_1, a_2, a_3, a_4) = \frac{1}{2} \log \left(1 + \left|\frac{a_2}{a_1}\right|^2 + \left|\frac{a_3}{a_1}\right|^2 + \left|\frac{a_4}{a_1}\right|^2\right) - \frac{1}{2} \log \left(1 + \left|\frac{a_2}{a_1}\right|^2\right).$$

We define $F_1 : \mathbb{C} \to \mathbb{P}^3$ and $F_2 : \mathbb{C} \to \mathbb{P}^1$ by

$$F_1(z) = [a_1 : a_2 : a_3 : a_4], \quad F_2(z) = [a_1 : a_2].$$

Let $H \subset \mathbb{P}^3$ be defined by $\{X_1 = 0\}$, where $[X_1 : X_2 : X_3 : X_4]$ is a homogeneous coordinate of $\mathbb{P}^3$. Let $H' \subset \mathbb{P}^1$ be defined by $\{Y_1 = 0\}$ where $[Y_1 : Y_2]$ is a homogeneous coordinate of $\mathbb{P}^1$. Then by the first main theorem (3.11), we have

$$\begin{aligned} A &= m(r, H, F_1) - m(1, H, F_1) + N(r, H, F_1) \le 4d \log r, \\ B &= m(r, H', F_2) - m(1, H', F_2) + N(r, H', F_2) \ge 0. \end{aligned}$$

By $N(r, H', F_2) \le N(r, H, F_1)$ we have

$$\begin{aligned} &\int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(re^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} \Lambda(a_1, a_2, a_3, a_4)(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= m(r, H, F_1) - m(1, H, F_1) - m(r, H', F_2) + m(1, H', F_2) \\ &\le A - B \le 4d \log r. \end{aligned}$$

and thus obtain (3.14). □

CHAPTER 4

Proof of Theorem 1.6

4.1. Beginning of the proof of Theorem 1.6

The first step in our proof of Theorem 1.6 is the estimate of oscillation of meromorphic functions on circles centered at the origin. For a meromorphic function f, we put

$$v(r,f,\theta) = \sup_{\tau\in[0,2\pi]} \left(\sup_{t\in[\tau,\tau+\theta]} \log|f(re^{it})| - \inf_{t\in[\tau,\tau+\theta]} \log|f(re^{it})| \right),$$

$$\lambda(r) = \min\left\{ 1, \left(\log^+ \frac{T(r)}{\log r} \right)^{-1} \right\}.$$

Proposition 4.1. *Let f be a transcendental meromorphic function in the complex plane. Let $\varepsilon > 0$. Then we have*

$$v(r,f,\lambda(r)^{20}) \le \varepsilon T(r,f)$$

for all $r > e$ outside a set of logarithmic density zero.

To prove this proposition, we begin with the following lemma:

Lemma 4.2. *Let f be a transcendental meromorphic function in the complex plane. Let $1 < \sigma < e$. Then we have*

$$\int_r^{\sigma r} \frac{v(t,f,(\log\sigma)^{10})}{t}\,dt < 647(\log\sigma)^2(T(\sigma^3 r,f)+c) \tag{4.1}$$

for $r > 1$, where c is a positive constant which only depends on f.

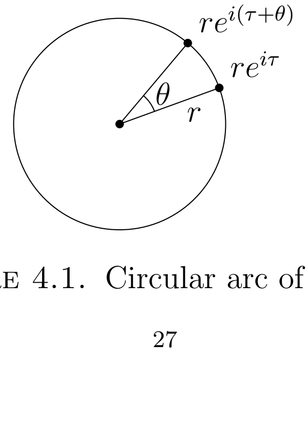

Figure 4.1. Circular arc of angle θ.

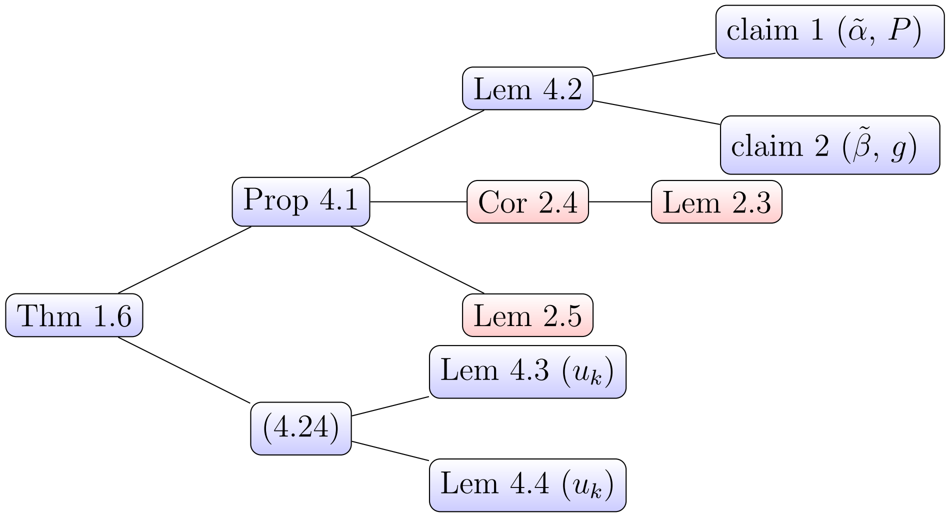

FIGURE 4.2. Structure of the proof of Theorem 1.6.

The structure of the proof of Theorem 1.6 is explained in Figure 4.2.

First, in Lemma 4.2, oscillation of $\log |f(re^{i\theta})|$ is estimated by oscillation of the Green function and Poisson kernel. Second, the function u_k is a generalization of the spherical derivative. By $\int_0^{2\pi} u_k(re^{i\theta})d\theta$, the important inequality (4.24) is established.

Compare with Figure 5.1, which explains the structure of the proof of Theorem 5.1.

PROOF. We apply the Poisson-Jensen formula ([**12**]). Let $P(z,\theta)$ be the Poisson kernel for the unit disc given by

$$P(z,\theta)=\frac{1-|z|^2}{|e^{i\theta}-z|^2},$$

and[1] let $g(z,a)$ be the Green function on the unit disc, namely

$$g(z,a)=\log\left|\frac{1-\overline{a}z}{z-a}\right|=-\log|\varphi_a(z)|\,,\quad \varphi_a(z)=\frac{a-z}{1-\overline{a}z}.$$

Suppose that $a_1,\dots,a_\mu$ are the zeros and that $b_1,\dots,b_\nu$ are the poles of $f(z)$ in $|z|<\rho$, where we put $\rho=\sigma^2 r$. We apply the Poisson-Jensen formula to obtain

(4.2)

$$\log|f(z)|=\int_0^{2\pi}\log|f(\rho e^{i\theta})|P\left(\frac{z}{\rho},\theta\right)\frac{d\theta}{2\pi}-\sum_{k=1}^{\mu}g\left(\frac{z}{\rho},\frac{a_k}{\rho}\right)+\sum_{k=1}^{\nu}g\left(\frac{z}{\rho},\frac{b_k}{\rho}\right)$$

for $|z|<\rho$.

In the following, we shall estimate the oscillation of $\log|f|$ in terms of the oscillations of the Poisson kernel and the Green functions.

For $0<t<1$, we put

$$\alpha(t,\theta,\tau)=\sup_{\tau_0\in[0,2\pi]}\left(\sup_{x\in[\tau_0,\tau_0+\tau]}P(te^{ix},\theta)-\inf_{x\in[\tau_0,\tau_0+\tau]}P(te^{ix},\theta)\right)$$

and

$$\beta(t,a,\tau)=\sup_{\tau_0\in[0,2\pi]}\left(\sup_{x\in[\tau_0,\tau_0+\tau]}g(te^{ix},a)-\inf_{x\in[\tau_0,\tau_0+\tau]}g(te^{ix},a)\right).$$

Set $\alpha(t,\tau)=\alpha(t,0,\tau)$. Since $P(z,\theta)=P(ze^{-i\theta},0)$, we have

$$\alpha(t,\theta,\tau)=\alpha(t,\tau). \tag{4.3}$$

---

[1]In [**19**], there is an extra factor $1/2\pi$ in the definition of $P(z,\theta)$. We can check our constants by a trivial example. If $f(z)\neq 0,\infty$ for $|z|\le\rho$, then (4.2) reduces to

$$\log|f(z)|=\int_0^{2\pi}\log|f(\rho e^{i\theta})|\frac{1-|z|^2}{|e^{i\theta}-z|^2}\frac{d\theta}{2\pi}.$$

If $f(z)\equiv e$, then we get

$$1=\log|f(0)|=\int_0^{2\pi}1\cdot\frac{1-0^2}{|e^{i\theta}-0|^2}\frac{d\theta}{2\pi}=\int_0^{2\pi}\frac{d\theta}{2\pi},$$

as it should be.

To visualize things, perhaps it is useful to calculate that

$$g\left(\frac{z}{\rho},\frac{a_k}{\rho}\right)=\log\left|\frac{1-\overline{(a_k/\rho)}(z/\rho)}{a/\rho-z/\rho}\right|=\log\left|\frac{\rho^2-\overline{a_k}z}{\rho(a_k-z)}\right|$$

and

$$P\left(\frac{z}{\rho},\theta\right)=\frac{1-|z/\rho|^2}{|e^{i\theta}-z/\rho|^2}=\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}.$$

Now by (4.2) and (4.3), for $0 < t < \rho$, we have

$$v(t,f,\tau) \le \alpha\left(\frac{t}{\rho},\tau\right)\int_0^{2\pi} |\log|f(\rho e^{i\theta})||\frac{d\theta}{2\pi} + \sum_{k=1}^{\mu}\beta\left(\frac{t}{\rho},\frac{a_k}{\rho},\tau\right) + \sum_{k=1}^{\mu}\beta\left(\frac{t}{\rho},\frac{b_k}{\rho},\tau\right).$$

Using the first main theorem, we obtain

$$v(t,f,\tau) \le \alpha\left(\frac{t}{\rho},\tau\right)(2T(\rho,f)+c) + \sum_{k=1}^{\mu}\beta\left(\frac{t}{\rho},\frac{a_k}{\rho},\tau\right) + \sum_{k=1}^{\mu}\beta\left(\frac{t}{\rho},\frac{b_k}{\rho},\tau\right),$$

where c is a positive constant which only depends on f. After dividing this estimate by t, we integrate the resulting estimate from ρ/σ^2 to ρ/σ to obtain

$$\int_{\rho/\sigma^2}^{\rho/\sigma} v(t,f,\tau)\frac{dt}{t} \le (2T(\rho,f)+c)\widetilde{\alpha}(\tau) + (\mu+\nu)\widetilde{\beta}(\tau),$$

where we put

$$\widetilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t,\tau)\frac{dt}{t}$$

and

$$\widetilde{\beta}(\tau) = \sup_{|a|\le 1}\int_{1/\sigma^2}^{1/\sigma} \beta(t,a,\tau)\frac{dt}{t}.$$

Using the definition of the counting function and the first main theorem, we have

$$\begin{aligned} \mu+\nu &= n(\rho,\infty,f) + n(\rho,0,f) \\ &\le \frac{1}{\log\sigma}(N(\sigma\rho,\infty,f) + N(\sigma\rho,0,f)) \\ &\le \frac{2}{\log\sigma}(T(\sigma\rho,f)+c), \end{aligned} \tag{4.4}$$

where c is a positive constant which only depends on f. Thus we obtain

$$\int_r^{\sigma r} v(t,f,\tau)\frac{dt}{t} \le \frac{1+\log\sigma}{\log\sigma}(2T(\sigma\rho,f)+c)(\widetilde{\alpha}(\tau)+\widetilde{\beta}(\tau)). \tag{4.5}$$

Claim 1. We have

$$\widetilde{\alpha}(\tau) \le \frac{4e^3\tau}{(\log\sigma)^2} \le \frac{4\cdot 27\tau}{(\log\sigma)^2}.$$

Proof of Claim 1. Since

$$\frac{\partial}{\partial\theta}P(te^{i\theta},0) = -\frac{2t(1+t)(1-t)\sin\theta}{|1-te^{i\theta}|^4},$$

we have

$$\left|\frac{\partial}{\partial\theta}P(te^{i\theta},0)\right| \le \frac{4}{(1-t)^3}, \quad 0<t<1.$$

Hence, we obtain

$$\alpha(t,\tau) \le \frac{4\tau}{(1-t)^3}.$$

Thus, we have

$$\widetilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t,\tau)\frac{dt}{t} \le \frac{4\tau\sigma^3}{(\sigma-1)^3}\log\sigma.$$

Since $\log\sigma \le \sigma - 1$ and[2] $\sigma^3 < e^3 < 27$, we complete the proof of the claim. ☐

Claim 2.

$$\widetilde{\beta}(\tau) \le \frac{(e^2+1)\tau}{(\log\sigma)^7} + 10e^2(\log\sigma)^3 \le \frac{10\tau}{(\log\sigma)^7} + 90(\log\sigma)^3.$$

Proof of Claim 2. We denote $\delta = (\log\sigma)^4$. For $|a|<1$, we set

$$\left[1/\sigma^2, 1/\sigma\right] = I(a) \cup J(a),$$

where

$$I(a) = \left[1/\sigma^2, 1/\sigma\right] \quad \cap \quad \left[|a|-\delta, |a|+\delta\right]$$
$$J(a) = \left[1/\sigma^2, 1/\sigma\right] \quad \setminus \quad \left[|a|-\delta, |a|+\delta\right].$$

Then we have

$$\int_{1/\sigma^2}^{1/\sigma} \beta(t,a,\tau)\frac{dt}{t} \le \underbrace{\sigma^2}_{<e^2<9}\int_{I(a)}\beta(t,a,\tau)dt + \int_{J(a)}\beta(t,a,\tau)\frac{dt}{t}. \tag{4.6}$$

First, we estimate the first term on the right hand side of (4.6). Since we have

$$\beta(t,a,\tau) \le \max_{\theta\in[0,2\pi]} g(te^{i\theta},a) = \log\frac{1-t|a|}{|t-|a||} \le \log\frac{1}{|t-|a||},$$

by

$$\left|\frac{1-\overline{a}z}{z-a}\right|^2 = 1 + \frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2},$$

we obtain

$$\int_{I(a)}\beta(t,a,\tau)dt \le 2\int_0^\delta \log\frac{1}{x}\,dx = 2(\log\sigma)^4 + 8(\log\sigma)^4\log\frac{1}{\log\sigma},$$

by

$$\int \log\frac{1}{x} = x + x\log\frac{1}{x}.$$

By

$$\log\frac{1}{x} \le \frac{1}{x}, \quad x\in(0,\infty).$$

[2] recall: $1<\sigma<e<3$

we have $\log(1/\log\sigma) \le (1/\log\sigma)$, and obtain

$$\int_{I(a)} \beta(t,a,\tau)\,dt \le 2(\log\sigma)^4 + 8(\log\sigma)^3 \le 10(\log\sigma)^3, \tag{4.7}$$

since $1 < \sigma < e$.

Next we estimate the second term on the right-hand side of (4.6). Since

$$\beta(t,a,\tau) = \beta(t,|a|,\tau),$$

it is enough to consider the case $0 < a < 1$. Since

$$\frac{\partial g}{\partial\theta}(te^{i\theta}) = \frac{at\sin\theta}{|1-ate^{i\theta}|^2} - \frac{at\sin\theta}{|a-te^{i\theta}|^2},$$

we have

$$\left|\frac{\partial g}{\partial\theta}(te^{i\theta})\right| \le \frac{1}{(1-at)^2} + \frac{1}{(a-t)^2} \le \frac{1}{(1-t)^2} + \frac{1}{(a-t)^2}.$$

Hence, we obtain

$$\beta(t,a,\tau) \le \frac{\tau}{(1-t)^2} + \frac{\tau}{(a-t)^2}.$$

Hence, on $t \in J(a)$, we have

$$\beta(t,a,\tau) \le \frac{\sigma^2\tau}{(\sigma-1)^2} + \frac{\tau}{(\log\sigma)^8}.$$

Since $\log\sigma \le \sigma - 1$, we obtain

$$\beta(t,a,\tau) \le \frac{\sigma^2\tau}{(\log\sigma)^2} + \frac{\tau}{(\log\sigma)^8}.$$

Thus, we obtain

$$\int_{J(a)} \beta(t,a,\tau)\frac{dt}{t} \le \frac{\sigma^2\tau}{\log\sigma} + \frac{\tau}{(\log\sigma)^7} \le \frac{(e^2+1)\tau}{(\log\sigma)^7} \le \frac{10\tau}{(\log\sigma)^7}. \tag{4.8}$$

From (4.6)–(4.8), we complete the proof of our claim. □

Now Lemma 4.2 is an obvious consequence of (4.5) and the claims above. (Recall that $\rho = \sigma^2 r$.)

Namely, set $\tau = (\log\sigma)^{10}$ and $\rho = \sigma^2 r$ to obtain

$$\widetilde{\alpha}(\tau) \le \frac{4e^3\tau}{(\log\sigma)^2} = 4e^3(\log\sigma)^8 \le 4e^3(\log\sigma)^3$$

and

$$\widetilde{\beta}(\tau) \le \frac{(e^2+1)\tau}{(\log\sigma)^7} + 10e^2(\log\sigma)^3 \le (e^2+1+10e^2)(\log\sigma)^3$$

and

$$\frac{1+\log\sigma}{\log\sigma}(2T(\sigma\rho,f)+c) \le \frac{4}{\log\sigma}(T(\sigma^3 r,f)+c/2) \le \frac{4}{\log\sigma}(T(\sigma^3 r,f)+c).$$

Hence, inequality (4.5) yields

$$\int_r^{\sigma r} v(t, f, (\log\sigma)^{10})\frac{dt}{t} \leq 4(4e^3 + e^2 + 1 + 10e^2)(\log\sigma)^2(T(\sigma^3 r, f) + \tilde{c})$$
$$\leq 647(\log\sigma)^2(T(\sigma^3 r, f) + c),$$

as desired.[3] □

Proof of Proposition 4.1 We apply Lemma 4.2 for $\sigma = \exp(\lambda(r)^2)$ to obtain

$$\int_r^{r\exp(\lambda(r)^2)} \frac{v(t, f, \lambda(r)^{20})}{t}\, dt < 647\lambda(r)^4 \left(T\left(e^{3\lambda(r)^2} r, f\right) + c\right).$$

By Corollary 2.4, we obtain

$$\int_r^{r\exp(\lambda(r)^2)} \frac{v(t, f, \lambda(r)^{20})}{t}\, dt < 647\lambda(r)^4(3T(r, f) + c)$$

outside a set of finite logarithmic measure.

Now given positive constants $\varepsilon > 0$ and $\varepsilon' > 0$, we have

$$\int_r^{r\exp(\lambda(r)^2)} \frac{v(t, f, \lambda(r)^{20})}{t}\, dt < \varepsilon\varepsilon'\lambda(r)^2 T(r, f)$$

outside a set $E_{\varepsilon\varepsilon'}$ of finite logarithmic measure. Set

$$F_\varepsilon = \left\{r \geq e;\, v(r, f, \lambda(r)^{20}) > \varepsilon T(r, f)\right\}.$$

Then we have

$$\int_{[r, re^{\lambda(r)^2}] \cap F_\varepsilon} \frac{dt}{t} \leq \int_r^{re^{\lambda(r)^2}} \frac{v(t, f, \lambda(r)^{20}}{\varepsilon T(t, f)t}\, dt$$
$$\leq \frac{1}{\varepsilon T(r, f)} \int_r^{re^{\lambda(r)^2}} \frac{v(t, f, \lambda(r)^{20}}{t}\, dt \leq \varepsilon'\lambda(r)^2$$

for all r outside $E_{\varepsilon,\varepsilon'}$. Thus, by Lemma 2.5, we establish Proposition 4.1. □

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[3]In the original paper [**19**], a factor 4 was missed and the corresponding estimate was $4(e^3 + e^2 + 1 + 10e^2) \leq 4(27 + 10 + 90) = 4 \cdot 127 = 508$.

Lemma 4.3. *Let f be a transcendental meromorphic function in the complex plane, and let k be a positive integer. Put*[4]

$$u_k = (k+1)\log^+|f| + \log|1/f^{(k)}|. \tag{4.9}$$

Then given a positive integer q, we have for all $r > 1$

(4.10)
$$\int_0^{2\pi} u_k(re^{i\theta})\frac{d\theta}{2\pi} \le \overline{m}_{k-1,q}(r,f) + (k-1)m(r,\infty,f) + v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r) + 2kq\log 3.$$

Proof. If f has a pole on the circle $|z| = r$, then $v(r,f,2\pi/q)$ is infinite. So the estimate is trivial. In the following, we show the estimate for r with the property that f does not have a pole on the circle $|z| = r$. We fix such r and work on the circle $|z| = r$.

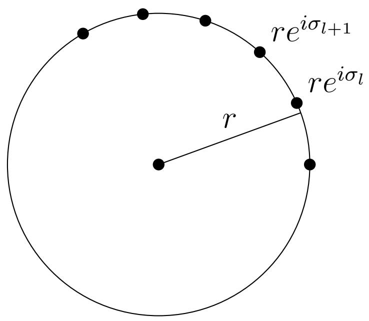


Figure 4.3. Circle divided to q arcs.

Set $\sigma_l = 2\pi l/q$. For $l = 0, 1, \dots, q-1$, we put

$$I_l = [\sigma_l, \sigma_{l+1}].$$

We define a polynomial $a_l(z)$ of degree less than k by

$$a_l(z) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(re^{i\sigma_l})(z - re^{i\sigma_l})^j.$$

Then we have

$$f^{(j)}(re^{i\sigma_l}) - a_l^{(j)}(re^{i\sigma_l}) = 0$$

[4] We note that

$$u_k = \log^+|f|^{k+1} + \log|1/f^{(k)}| \le \log(1+|f|^{k+1}) + \log|1/f^{(k)}| = -\log\frac{|f^{(k)}|}{1+|f|^{k+1}}.$$

For u_k replaced by $-\log f^{\#}$, Lemma 4.3 reads

$$\int_0^{2\pi} \log f^{\#}((re^{i\theta}))\frac{d\theta}{2\pi} \le \bar{m}_q(r,f) + v(r,f^{\#},2\pi/q) + \log(2\pi r/q),$$

see Lemma 5.6.

for $0 \leq j \leq k-1$. Thus, we have

$$f(re^{i\theta})-a_l(re^{i\theta}) = \int_{\sigma_l}^{\theta}\int_{\sigma_l}^{\theta_1}\cdots\int_{\sigma_l}^{\theta_{k-1}} f^{(k)}(re^{i\theta_k}))d(re^{i\theta_k})\cdots d(re^{i\theta_2})d(re^{i\theta_1}).$$

Thus, for $\theta \in I_l$, we have

$$|f(re^{i\theta}) - a_l(re^{i\theta})| \leq e^{\tau_l}(2\pi r)^k,$$

where we put

$$\tau_l = \max_{s\in I_l} \log |f^{(k)}(re^{is})|.$$

Since we have

$$\log \frac{1}{|f^{(k)}(re^{i\theta})|} \leq -\tau_l + v(r, f^{(k)}, 2\pi/q)$$

for $\theta \in I_l$, we obtain

$$\log \frac{1}{|f^{(k)}(re^{i\theta})|} \leq \log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} + v(r, f^{(k)}, 2\pi/q) + k\log(2\pi r)$$

for $\theta \in I_l$. Hence, we obtain

$$\int_0^{2\pi} \log \frac{1}{|f^{(k)}(re^{i\theta})|}\frac{d\theta}{2\pi} \leq \sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|}\frac{d\theta}{2\pi} + v(r, f^{(k)}, 2\pi/q) + k\log(2\pi r).$$

Thus, using

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})|\frac{d\theta}{2\pi} \leq m(r,\infty,f),$$

we have

(4.11)

$$\int_0^{2\pi} u_k(re^{i\theta})\frac{d\theta}{2\pi} \leq \sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} + 2\log^+|f(re^{i\theta})|\right)\frac{d\theta}{2\pi} + (k-1)m(r,\infty,f) + v(r, f^{(k)}, 2\pi/q) + k\log(2\pi r).$$

We estimate the right-hand side of (4.11).

Claim 1. Let $a(z)$ be a polynomial of degree less than k. Then we have

$$\frac{\log^+|a(re^{i\sigma_l})|}{q} \leq \int_{\sigma_l}^{\sigma_{l+1}} \log^+|a(re^{i\theta})|\frac{d\theta}{2\pi} + 2k\log 3.$$

Proof. It is enough to prove this claim assuming $a(z) \neq 0$. We consider the following function:

$$U(a(z)) = \frac{\log|a(e^{i\sigma_0})|}{q} - \int_{\sigma_0}^{\sigma_1} \log|a(e^{i\theta})|\frac{d\theta}{2\pi}.$$

Then we have

$$U(\lambda a(z)) = U(a(z)) \tag{4.12}$$

for non-zero λ. If

$$a(z) = \alpha_0(z - \alpha_1)\cdots(z - \alpha_{k'}), \quad (k' < k),$$

then we have

$$U(a(z)) = U(z - \alpha_1) + \cdots + U(z - \alpha_{k'}). \tag{4.13}$$

Now we observe that

$$U(z - a) \leq 2\log 3. \tag{4.14}$$

Indeed if $|\alpha| \leq 2$, then we have

$$\frac{\log|e^{i\sigma_0} - \alpha|}{q} \leq \frac{\log 3}{q}$$

and

$$-\int_{\sigma_0}^{\sigma_1} \log|e^{i\theta} - \alpha|\frac{d\theta}{2\pi} = -\log^+|\alpha| + \int_{[0,2\pi]\setminus[\sigma_0,\sigma_1]} \log|e^{i\theta} - \alpha|\frac{d\theta}{2\pi} < \log 3.$$

This shows (4.14).

Next we consider the other case $|\alpha| > 2$. By (4.12), we have

$$U(z - \alpha) = U(z/\alpha - 1).$$

Using

$$\log\frac{1}{2} < \log|e^{i\theta}/\alpha - 1| < \log\frac{3}{2}$$

we obtain (4.14). Thus we have proved (4.14).

Combining (4.13) and (4.14), we obtain

$$U(a(z)) < 2k\log 3.$$

Now for a polynomial $a(z) \neq 0$ of degree less than k, we consider the polynomial $b(z) = a(re^{i\sigma_l}z)$. Then we have

$$\frac{\log|a(re^{i\sigma_l})|}{q} - \int_{\sigma_l}^{\sigma_{i+1}} \log|a(re^{i\theta})|\frac{d\theta}{2\pi} = \frac{\log|b(e^{i\sigma_0})|}{q} - \int_{\sigma_0}^{\sigma_1} \log|b(e^{i\theta})|\frac{d\theta}{2\pi}.$$

Hence, we have

$$\begin{aligned} \frac{\log|a(re^{i\sigma_l})|}{q} &\leq \int_{\sigma_l}^{\sigma_{l+1}} \log|a(re^{i\theta})|\frac{d\theta}{2\pi} + 2k\log 3 \\ &\leq \int_{\sigma_l}^{\sigma_{l+1}} \log^+|a(re^{i\theta})|\frac{d\theta}{2\pi} + 2k\log 3. \end{aligned} \tag{4.15}$$

Our claim is an obvious consequence of this estimate. □

Claim 2. Let a and b be two points in $\mathbb{C}$. Then we have

$$\log\frac{1}{|a - b|} + \log^+|a| + \log^+|b| \leq \log\frac{1}{[a,b]}. \tag{4.16}$$

Proof. Since

$$[a,b] = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}},$$

we obtain

$$\log \frac{1}{[a,b]} = \log \frac{1}{|a-b|} + \log \sqrt{1+|a|^2} + \log \sqrt{1+|b|^2}$$
$$\geq \log \frac{1}{|a-b|} + \log^+ |a| + \log^+ |b|.$$

□

We go back to the proof of Lemma 4.3. For $\theta \in I_l$, we have

$$\begin{aligned} \log^+ |f(re^{i\theta})| &\leq \log^+ |f(re^{i\sigma_l})| + v(r,f,2\pi/q) \\ &= \log^+ |a_l(re^{i\sigma_l})| + v(r,f,2\pi/q). \end{aligned} \tag{4.17}$$

Hence, we obtain

$$\begin{aligned} &\int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} + 2\log^+ |f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\ &\leq \int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} + \log^+ |f(re^{i\theta})| + \log^+ |a_l(re^{i\theta})|| \right) \frac{d\theta}{2\pi} \\ &\quad + \frac{v(r,f,2\pi/q)}{q} + 2k\log 3. \end{aligned}$$

We use the two claims above to obtain

$$\begin{aligned} &\int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} + 2\log^+ |f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\ &\leq \int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f(re^{i\theta}), a_l(re^{i\theta})]} \frac{d\theta}{2\pi} + \frac{v(r,f,2\pi/q)}{q} + 2k\log 3. \end{aligned}$$

Combining this estimate with (4.11), we obtain

$$\begin{aligned} \int_0^{2\pi} u_k(re^{i\theta}) \frac{d\theta}{2\pi} \leq &\sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{[f(re^{i\theta}) - a_l(re^{i\theta})]} \right) \frac{d\theta}{2\pi} \\ &+ (k-1)m(r,\infty,f) \\ &+ v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r) + (2k\log 3)\cdot q. \end{aligned}$$

Now since

$$\sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \left(\log \frac{1}{[f(re^{i\theta}) - a_l(re^{i\theta})]} \right) \frac{d\theta}{2\pi} \leq \overline{m}_{k-1,q}(r,f),$$

we complete the proof of Lemma 4.3. □

LEMMA 4.4. *We have*[5]

$$\int_0^{2\pi} u_k(re^{i\theta})\frac{d\theta}{2\pi} = (k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f) + O(1). \tag{4.18}$$

PROOF. Put

$$\widetilde{u} = (k+1)\log|f| + \log|1/f^{(k)}|.$$

Then we have

$$\int_0^{2\pi} \widetilde{u}(re^{i\theta})\frac{d\theta}{2\pi} = (k+1)\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(re^{i\theta})}\right|\frac{d\theta}{2\pi}.$$

By the first main theorem, we have

$$\begin{aligned}\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{2\pi} &= N(r,0,f) - N(r,\infty,f),\\ \int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(re^{i\theta})}\right|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(e^{i\theta})}\right|\frac{d\theta}{2\pi} &= N(r,\infty,f^{(k)}) - N(r,0,f^{(k)}).\end{aligned} \tag{4.19}$$

Combining these estimates with

$$N(r,\infty,f^{(k)}) = N(r,\infty,f) + k\overline{N}(r,\infty,f),$$

we obtain

$$\begin{aligned}\int_0^{2\pi} \widetilde{u}(re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} \widetilde{u}(e^{i\theta})\frac{d\theta}{2\pi} &= (k+1)(N(r,0,f) - N(r,\infty,f)) + N(r,\infty,f^{(k)})\\ &\quad - N(r,0,f^{(k)})\\ &= (k+1)N(r,0,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f).\end{aligned} \tag{4.20}$$

We note that

$$u_k = \widetilde{u} + (k+1)\log^+|1/f|. \tag{4.21}$$

By the first main theorem, we have

$$\int_0^{2\pi} \log^+|1/f(re^{i\theta})|\frac{d\theta}{2\pi} + N(r,0,f) = T(r,f) + O(1). \tag{4.22}$$

Now, by (4.20)-(4.22), we obtain

$$\int_0^{2\pi} u_k(re^{i\theta})\frac{d\theta}{2\pi} = (k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f) + O(1).$$

Thus Lemma 4.4 is proved. □

[5]For u_k replaced by $\log f^{\#}$, we have (4.18) in the form

$$\int_0^{2\pi} \log f^{\#}((re^{i\theta}))\frac{d\theta}{2\pi} = -2T(r,f) + N_1(r,f) + \int_0^{2\pi} \log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi}.$$

See Lemma 2.8, that is, [**2**, Proposition 2.4.2].

By Lemmas 4.3 and 4.4, we obtain

$$
\begin{aligned}
&(k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f)\\
&\le \overline{m}_{k-1,q}(r,f) + (k-1)m(r,\infty,f)\\
&+ v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q)\\
&+ k\log(2\pi r) + 2kq\log 3 + C,
\end{aligned} \tag{4.23}
$$

where C is a positive constant which only depends on f. By the first main theorem, we have

$$T(r,f) = m(r,\infty,f) + \overline{N}(r,\infty,f) + N_1(r,\infty,f) + O(1).$$

Hence, we obtain

(4.24)

$$
\begin{aligned}
2T(r,f) + (k-1)\overline{N}(r,\infty,f) &\le \overline{m}_{k-1,q}(r,f) + N(r,0,f^{(k)}) + N_1(r,\infty,f)\\
&+ v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r)\\
&+ 2kq\log 3 + C.
\end{aligned}
$$

Proof of Theorem 1.6. Let f be a transcendental meromorphic function and let $\varepsilon > 0$. By Proposition 4.1, we have

$$v(r,f,\lambda(r)^{20}) < \frac{\varepsilon}{21}T(r,f)$$

outside some exceptional set of logarithmic density zero. For r sufficiently large, we have $2\pi/\nu(r) < 7\lambda(r)^{20}$. Hence, (since $\theta \mapsto v(r,f,\theta)$ is increasing), we have

$$v(r,f,2\pi/\nu(r)) < \frac{\varepsilon}{3}T(r,f) \tag{4.25}$$

for all $r > e$ outside of a set E_1 of logarithmic density zero. Again by Proposition 4.1, we have

$$v\left(r,f^{(k)},\widetilde{\lambda}(r)^{20}\right) < \frac{\varepsilon}{42(k+2)}T(r,f^{(k)})$$

for all $r > e$ outside some set E_2 of logarithmic density zero, where we set

$$\widetilde{\lambda}(r) = \min\left\{1, \left(\log^+ \frac{T(r,f^{(k)})}{\log r}\right)^{-1}\right\}.$$

By Nevanlinna's Lemma of the logarithmic derivative, we have

$$T(r,f^{(k)}) \le (k+2)T(r,f)$$

for all $r > e$ outside a set E_3 of finite linear measure. Hence, we obtain

$$v\left(r,f^{(k)},\widetilde{\lambda}(r)^{20}\right) < \frac{\varepsilon}{21}T(r,f)$$

for all $r > e$ outside the set $E_2 \cup E_3$ of logarithmic density zero. We find a positive constant r_0 such that $\lambda(r)^{20} < 2\widetilde{\lambda}(r)^{20}$ for $r > r_0$ outside E_3. Hence, we have

$$v\left(r, f^{(k)}, \lambda(r)^{20}\right) < \frac{\varepsilon}{21} T(r, f) \tag{4.26}$$

for $r > e$ outside an exceptional set E_4 of logarithmic density zero.

Since f is transcendental, we find a positive constant r_1 such that

$$k \log(2\pi r) + 2k\nu(r) \log 3 + C < \frac{\varepsilon}{3} T(r, f) \tag{4.27}$$

for $r > r_1$.

Now we put

$$E = [e, r_1] \cup E_1 \cup E_4.$$

Then E has logarithmic density zero. By (4.25)-(4.27), we have

$$v\left(r, f, \frac{2\pi}{\nu(r)}\right) + v\left(r, f^{(k)}, \frac{2\pi}{\nu(r)}\right) + k \log(2\pi r) + 2k\nu(r) \log 3 + c < \varepsilon T(r, f)$$

for all $r > e$ outside E. Combining this estimate with (4.24), we complete[6] the proof of Theorem 1.6. □

[6] end of 5^{th} seminar in 2019

CHAPTER 5

Finite order functions and asymptotic SMT

This chapter follows the manuscript "K. Yamanoi - *On a reversal of the second main theorem for meromorphic functions of finite order*".

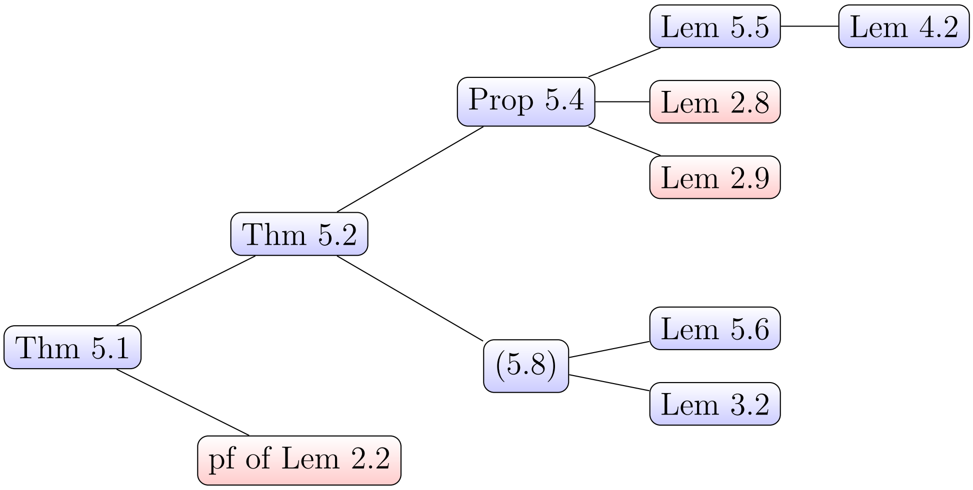


FIGURE 5.1. Structure of the proof of Theorem 5.1.

The structure of the proof of Theorem 5.1 is explained in Figure 5.1.

First, Lemma 4.2 is restated Lemma 5.5, a special case with $\sigma = 2$. Lemma 2.8 yields $T(8r, f) \leq C(\varepsilon, \rho(f))T(r, f)$.

Second, the important inequality (5.8) is proved. Inequality (5.8) is a special case of (4.24). Now, we are dealing with f' instead of $f^{(k)}$. Therefore, we estimate the spherical derivative $f^{\#} = |f'|/(1 + |f|^2)$ instead of the function u_k, which was needed in the proof of (4.24).

Compare with Figure 4.2, which explains the structure of the proof of Theorem 1.6.

THEOREM 5.1. *Let f be a transcendental meromorphic function of finite order. Let $\nu : \mathbb{R}_{>e} \to \mathbb{N}_{>0}$ satisfy $\nu(r) \to \infty$ and $\log \nu(r) = o(T(r, f))$ as $r \to \infty$. Then we have*

$$\bar{m}_{\nu(r)}(r, f) + N_1(r, f) = 2T(r, f) + o(T(r, f)) \tag{5.1}$$

where $r \to \infty$ outside a set of logarithmic density 0.

By Theorem 3.3, (5.1) holds for general transcendental meromorphic functions, including the case of infinite order, provided that the function ν satisfies the growth condition (1.7). Theorem 5.1 shows that $\nu(r)$ may be arbitrary slow growth if f is of finite order.

If $\nu : \mathbb{R}_{>e} \to \mathbb{N}_{>0}$ satisfies $\log \nu(r) = o(T(r, f))$, then Theorem 3.1 says that

$$\bar{m}_{\nu(r)}(r, f) + N_1(r, f) \leq 2T(r, f) + o(T(r, f)) \tag{5.2}$$

for all $r > e$ outside an exceptional set of finite linear measure, and the following theorem. Thus the issue is to prove the reversal of (5.2). This is contained in the following theorem.

THEOREM 5.2. *Let f be a transcendental meromorphic function of finite order λ. For $0 < \varepsilon < 1$, there exists a positive integer $q_{\lambda,\varepsilon}$ and a set $E_{f,\varepsilon} \subset [e, \infty)$ with*

$$\overline{\log \operatorname{dens}}\, E_{f,\varepsilon} < \varepsilon$$

such that for all $r \geq e$ outside $E_{f,\varepsilon}$, the following inequality holds:

$$2T(r, f) \leq \bar{m}_{q_{\lambda,\varepsilon}}(r, f) + N_1(r, f) + \varepsilon T(r, f).$$

Here $q_{\lambda,\varepsilon}$ depends only on λ and ε.

The proof of Theorem 5.2 shows that we may take $q_{\lambda,\varepsilon} = \lceil 2^{203} 2^{7680\lambda/\varepsilon^2} \rceil$, where $\lceil x \rceil$ is the smallest integer which is not less than x.

REMARK 5.3. Let $a_1, \dots, a_q \in \overline{\mathbb{C}}$ be distinct points. We have

$$\sum_{i=1}^{q} m(r, a_i, f) = \int_0^{2\pi} \max_{1 \leq i \leq q} \log \frac{1}{[f((re^{i\theta})), a_i]} \frac{d\theta}{2\pi} + O(1) \leq \bar{m}(r, f) + O(1),$$

where $O(1)$ only depends on $a_1, \dots, a_q$. Thus we may recover usual estimate of Nevanlinna's second main theorem

$$\sum_{i=1}^{q} m(r, a_i, f) + N_1(r, f) \leq 2T(r, f) + o(T(r, f)) \tag{5.3}$$

from 5.2, provided $\nu(r) \to \infty$ as $r \to \infty$.

The question of reversal of (5.3) is already discussed in [**12**] and [**16**, Chapter 4]. For many familiar functions, (5.3) is known to be an asymptotic equality rather than inequality. For instance, this holds for meromorphic functions with finitely many critical and asymptotic values, provided $\{a_1, \dots, a_q\}$ contains all critical and asymptotical values (cf. [**14**]). See also [**3**] for other investigations of this problem from potential-theoretic view point.

For a meromorphic function f, we put

$$v(r, f, \theta) = \sup_{\tau} \left(\sup_{t \in [\tau, \tau+\theta]} \log |f((re^{i\theta}))| - \inf_{t \in [\tau, \tau+\theta]} \log |f((re^{i\theta}))| \right).$$

We first show

Proposition 5.4. *Let f be a transcendental meromorphic function of finite order λ. Let $0 < \varepsilon < 1$. Then there exists a positive constant $\theta_{\lambda,\varepsilon}$ such that*

$$v(r, f, \theta_{\lambda,\varepsilon}) \leq \varepsilon T(r, f)$$

for all $r > e$ outside an exceptional set $E_{f,\varepsilon}$ with $\overline{\log \mathrm{dens}}\, E_{f,\varepsilon} < \varepsilon$.

The proof of Proposition 5.4 shows that we may take $\theta_{\lambda,\varepsilon} = \varepsilon^{20}/2^{140}2^{120\lambda/\varepsilon^2}$.

To prove Proposition 5.4, we need several lemmas.

Lemma 5.5. *For $0 < \varepsilon < 1$, there exists $\tau_\varepsilon > 0$ such that*

$$\int_r^{2r} \frac{v(t, f, \tau_\varepsilon)}{t} dt < \varepsilon T(8r, f)$$

for $r > r_0$, where $r_0 > 1$ is a constant which only depends on f.

The proof shows that we may take $\tau_\varepsilon = \varepsilon^{10}/2^{110}$.

Proof. By Lemma 4.2, we have the following: Let $1 < \sigma < e$. Then

$$\int_r^{\sigma r} \frac{v(t, f, (\log \sigma)^{10})}{t} dt < 647(\log \sigma)^2 (T(\sigma^3 r, f) + c), \tag{5.4}$$

for $r > 1$, where c is a positive constant which only depends on f.

Now given $0 < \varepsilon < 1$, we take a positive integer l such that

$$l \geq \frac{2 \cdot 647(\log 2)^2}{\varepsilon} = \frac{1294(\log 2)^2}{\varepsilon}.$$

We take $r_0 > 1$ such that $T(r_0, f) > c$. Then for $i = 0, \dots, l-1$ and $r > r_0$, (5.4) yields

$$\begin{aligned}\int_{2^{i/l} r}^{2^{(i+1)/l} r} \frac{v(t, f, (\log 2^{1/l})^{10})}{t} dt &< 1294(\log 2^{1/l})^2 T(2^{(3+i)/l} r, f) \\ &\leq \frac{1294(\log 2)^2}{l^2} T(8r, f).\end{aligned}$$

Thus we obtain

$$\int_r^{2r} \frac{v(t, f, (\log 2^{1/l})^{10})}{t} dt < \varepsilon T(8r, f)$$

for $r > r_0$. We set $\tau_\varepsilon = (\log 2^{1/l})^{10}$ to conclude the proof. □

For the constants, we have

$$\frac{1}{l} \leq \frac{\varepsilon}{1294(\log 2)^2}$$

and

$$\frac{1}{l^{10}} \leq \frac{\varepsilon^{10}}{1294^{10}(\log 2)^{20}}.$$

Therefore

$$\tau_\varepsilon = (\log 2^{1/l})^{10} = \frac{1}{l^{10}}(\log 2)^{10} \leq \frac{\varepsilon^{10}}{1294^{10}(\log 2)^{10}}.$$

Now,

$$1294^{10}(\log 2)^{10} \leq \frac{1}{3.36 \cdot 10^{29}} \leq \frac{1}{2^{98}}.$$

Hence, we can take $\tau_\varepsilon = \varepsilon^{10}/2^{98}$. Of course, the claim is true for a smaller constant $\tau_\varepsilon = \varepsilon^{10}/2^{110}$ (Yamanoi's original constant here.)

In order to deal with the term $T(8r, f)$, we use the Borel-type lemmas Lemma 2.8 and Lemma 2.9. (In the beginning of the seminar in spring 2019, we proved a few Borel-type lemmas. Now, we need two more such results.)

Proof of Proposition 5.4. Let $0 < \varepsilon < 1$. First we apply Lemma 5.5 for

$$\frac{\varepsilon^2/4}{C(\varepsilon^2/2)},$$

where $C(\varepsilon^2/2) = 2 \cdot 8^{4\lambda/\varepsilon^2}$ is the constant from Lemma 2.8. Then we obtain a positive constant $\theta_{\lambda,\varepsilon}$ such that

$$\int_r^{2r} \frac{v(t, f, \theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon^2/4}{C(\varepsilon^2/2)} T(8r, f)$$

for $r > r_0$. Here $\theta_{\lambda,\varepsilon} = \tau(\varepsilon^2/2^{3+(12\lambda/\varepsilon^2)})$.

Next we apply Lemma 2.8 for $\varepsilon^2/2$ to obtain a set E such that

$$T(8r, f) < C(\varepsilon^2/2)T(r, f)$$

for all r outside E. Here we have

$$\overline{\log \mathrm{dens}}\, E < \frac{\varepsilon^2}{2}.$$

Thus we have

$$\int_r^{2r} \frac{v(t, f, \theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon^2}{4} T(r, f)$$

for all $r > r_0$ outside E.

Now we set

$$F = \{r\,;\, v(r, f, \theta_{\lambda,\varepsilon}) \geq \varepsilon T(r, f)\}.$$

Then we have

$$\begin{aligned}\int_{[r,2r]\cap F} \frac{dt}{t} &\leq \int_r^{2r} \frac{v(t, f, \theta_{\lambda,\varepsilon})}{\varepsilon T(t, f)} \frac{dt}{t} \\ &\leq \frac{1}{\varepsilon T(r, f)} \int_r^{2r} \frac{v(t, f, \theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon}{4}\end{aligned}$$

for all $r > r_0$ outside E. Thus by Lemma 2.9, we have

$$\overline{\log \mathrm{dens}}\, F < \frac{\varepsilon}{4 \log 2} + \frac{\varepsilon^2}{2} < \frac{\varepsilon}{2 \log e} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude the proof of Proposition 5.4. □

LEMMA 5.6. *Let f be a non-constant meromorphic function and $q > 0$ a positive integer. Then*

$$\int_0^{2\pi} \log \frac{1}{f^{\#}((re^{i\theta}))} \frac{d\theta}{2\pi} \le \bar{m}_q(r,f) + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q). \tag{5.5}$$

PROOF. Let $\sigma_k = 2\pi k/q$. For $l = 0, 1, \ldots, q-1$, we set $I_l = [\sigma_l, \sigma_{l+1}]$ and $a_l = f(re^{i\sigma_l})$.

In general

$$[a,b] \le d_{\text{spherical}}(a,b) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1+|z|^2} \le \int_{\gamma} \frac{|dz|}{1+|z|^2},$$

for any piecewise smooth $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = a, \gamma(1) = b$. Therefore, we have

$$\big[f((re^{i\theta})), a_l\big] \le \int_{\sigma_l}^{\theta} f^{\#}((re^{i\theta}))\, r d\theta.$$

Set

$$\tau_l = \max_{s \in I_l} \log f^{\#}(re^{is}).$$

Then for $\theta \in I_l$, we have

$$\big[f((re^{i\theta})), a_l\big] \le e^{\tau_l} 2\pi r/q. \tag{5.6}$$

In terms of $v(r, f^{\#}, \theta)$, for $\theta \in I$ we have

$$\log \frac{1}{f^{\#}((re^{i\theta}))} \le -\tau_l + v(r, f^{\#}, 2\pi/q).$$

Combining this estimate with (5.6), we obtain

$$\log \frac{1}{f^{\#}((re^{i\theta}))} \le \log \frac{1}{[f((re^{i\theta})), a_l]} + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q)$$

for $\theta \in I_l$. Thus

$$\int_0^{2\pi} \log \frac{1}{f^{\#}((re^{i\theta}))} \frac{d\theta}{2\pi} \le \sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f((re^{i\theta})), a_l]} \frac{d\theta}{2\pi} + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q).$$

By

$$\sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f((re^{i\theta})), a_l]} \frac{d\theta}{2\pi} \le \bar{m}(r,f),$$

we conclude (5.5) □

We recall Lemma 3.2, which states that for a transcendental meromorphic function f on the complex plane, we have

(5.7)
$$\int_0^{2\pi} \log f^{\#}((re^{i\theta}))\frac{d\theta}{2\pi} = -2T(r,f) + N_1(r,f) + \int_0^{2\pi} \log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi}.$$

Now we prove Theorem 5.2. Let $q > 0$ be a positive integer. By (5.5), (5.7) and

$$v(r, f^{\#}, 2\pi/q) \leq 2v(r,f,2\pi/q) + v(r,f',2\pi/q),$$

we have[1]

$$\begin{aligned} 2T(r,f) &\leq \bar{m}_q(r,f) + N_1(r,f) \\ &\quad + 2v(r,f,2\pi/q) + v(r,f',2\pi/q) + \log r + C \end{aligned} \tag{5.8}$$

for all $r > 1$, where C is a positive constant which only depends on f.

Now let $0 < \varepsilon < 1$. Set $q_{\lambda,\varepsilon} = \lceil 2\pi/\theta_{\lambda,\varepsilon/8}\rceil$. By Proposition 5.4, we have

$$v(r,f,2\pi/q_{\lambda,\varepsilon}) < \frac{\varepsilon}{8}T(r,f) \tag{5.9}$$

for all $r > e$ outside E_1 with

$$\overline{\log \text{dens}}\, E_1 < \frac{\varepsilon}{8}. \tag{5.10}$$

Since f' has the same order λ, Proposition 5.4 yields that

$$v(r,f',2\pi/q_{\lambda,\varepsilon}) < \frac{\varepsilon}{8}T(r,f')$$

for all $r > e$ outside E_2 with

$$\overline{\log \text{dens}}\, E_2 < \frac{\varepsilon}{8}.$$

By Nevanlinna's Lemma on logarithmic derivative, we have

$$T(r,f') \leq \frac{5}{2}T(r,f)$$

for all $r > e$ outside E_3 of finite linear measure. Hence we obtain

$$v(r,f',2\pi/q_{\lambda,\varepsilon}) < \frac{5\varepsilon}{16}T(r,f) \tag{5.11}$$

for $r > e$ and $r \notin E_2 \cup E_3$, where we have

$$\overline{\log \text{dens}}(E_2 \cup E_3) < \frac{\varepsilon}{8}. \tag{5.12}$$

Since f is transcendental, we find a positive constant r_1 such that

$$\log r + C < \frac{\varepsilon}{8}T(r,f) \tag{5.13}$$

for $r > r_1$.

Now we put

$$E = [e, r_1] \cup E_1 \cup E_2 \cup E_3.$$

[1] A more general inequality is (4.23).

Then by (5.10) and (5.12), we have

$$\overline{\log \operatorname{dens}} E < \varepsilon.$$

By (5.9), (5.11), (5.13), we have

$$2v(r, f, 2\pi/q_{\lambda,\varepsilon}) + v(r, f', 2\pi/q_{\lambda,\varepsilon}) + \log r + C < \varepsilon T(r, f)$$

for all $r > e$ outside E. Combining this estimate with (5.8), we conclude the proof of Theorem 5.2. □

We prove Theorem 5.1. Let n be a positive integer. We recall $E(f, 1/2^n)$ and $q(\lambda, 1/2^n)$ from Theorem 5.2. By $\nu(r) \to \infty$ as $r \to \infty$, we may take $c_n > e$ such that $\nu(r) > q(\lambda, 1/2^n)$ for all $r > c_n$. We define $F(1/2^n) \subset [e, \infty)$ such that $r \in F(1/2^n)$ iff

$$2T(r, f) > \bar{m}_{\nu(r)}(r, f) + N_1(r, f) + \frac{1}{2^n} T(r, f).$$

Then we have

$$F(1/2^n) \cap [c_n, \infty) \subset E(f, 1/2^n) \cap [c_n, \infty).$$

Thus we have $\overline{\log \operatorname{dens}} F(1/2^n) < 1/2^n$.

Now we take $r_n > e$ such that

$$\frac{\int_{F(1/2^n) \cap [e, r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n},$$

for all $r \geq r_n$. We may assume without loss of generality that the sequence $r_1, r_2, \dots$ satisfies $r_1 < r_2 < r_3 < \cdots$ and $r_n \to \infty$ as $n \to \infty$. For $r \in [r_n, r_{n+1})$, we set $\varepsilon(r) = 1/2^n$. Then $\varepsilon(r)$ is defined for all $r \geq r_1$ and $\varepsilon(r) \to 0$ as $r \to \infty$.

We define $F \subset [r_1, \infty)$ such that $r \in F$ iff

$$2T(r, f) > \bar{m}_{\nu(r)}(r, f) + N_1(r, f) + \varepsilon(r) T(r, f).$$

Then we have

$$F \cap [r_1, r_{n+1}) \subset F(1/2^n) \cap [r_1, r_{n+1}].$$

Thus we have

$$\frac{\int_{F \cap [r_1, r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n}$$

for $r_n \leq r < r_{n+1}$. Thus F has logarithmic density 0. Thus we have

$$2T(r, f) \leq \bar{m}_{\nu(r)}(r, f) + N_1(r, f) + o(T(r, f))$$

where $r \to \infty$ outside a set of logarithmic density 0. Combining this estimate with (5.2), we conclude the proof of Theorem 5.1. □

CHAPTER 6

General form of Theorem 1.4 and its local version

In the rest of this paper, we shall prove Theorem 1.7 in the following general form.

THEOREM 6.1. *Let f be a transcendental meromorphic function in the plane and let d be a positive integer. Then there exists a set $E_{f,d} \subset \mathbb{R}_{>0}$ of finite linear measure with the following property: Given an arbitrary q-tuple of distinct $a_1, \ldots, a_q \in \mathcal{R}_d$ and an arbitrary $\varepsilon > 0$, we have*

$$\begin{aligned} &\int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{1\le j\le q} N_1(r, a_j, f) \\ &\qquad \le (2+\varepsilon)T(r,f) + \frac{q^{17}}{\varepsilon^4} T(r)^{4/5} (\log r)^{1/5} \end{aligned} \tag{6.1}$$

for all $r > 0$ outside $E_{f,d}$.

Derivation of Theorem 1.7 from Theorem 6.1 Let $a_1, \ldots, a_p$ be distinct points in the Riemann sphere. Let $b_1, \ldots, b_q \in \mathcal{R}_d$. We apply Theorem 6.1 to the subset

$$\{a_1, \ldots, a_p, b_1, \ldots, b_q\} \subset \mathcal{R}_d.$$

Then we obtain, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} &\int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), b_j(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{1\le i\le p} N_1(r, a_i, f) \\ &\qquad \le (2+\varepsilon)T(r,f) + \frac{q^{17}}{\varepsilon^4} T(r)^{4/5} (\log r)^{1/5} \end{aligned} \tag{6.2}$$

for all $r > 0$ outside $E_{f,d}$. Taking the supremum for $(b_1, \ldots, b_q)$ on the left-hand side, we obtain

$$\overline{m}_{d,q}(r,f) + \sum_{1\le i\le p} N_1(r, a_i, f) \le\le (2+\varepsilon)T(r,f) + \frac{q^{17}}{\varepsilon^4} T(r)^{4/5} (\log r)^{1/5}$$

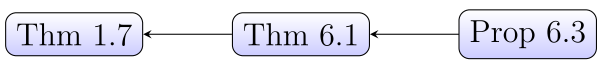


FIGURE 6.1. Theorem 1.7 has a more general version Theorem 6.1, which in turn has a local version Proposition 6.3.

for all $r > 0$ outside of $E_{f,d}$, as desired. □

We introduce a local version of Theorem 6.1. Some notation are needed.

DEFINITION 6.2. We denote by γ_d a constant such that $\gamma_d > e$ so that the following estimates hold for all $r > \gamma_d$:

(1) $\log r \leq T(r)$,
(1) $T(r, \mathrm{cr}(f, a_i, a_j, a_k)) \leq 2T(r)$ for all distinct $a_i, a_j, a_k \in \mathcal{R}_d$.

Here cr denotes the cross-ratio:

$$\mathrm{cr}(w_1, w_2, w_3, w_4) = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}.$$

Note that by (1.11), that is,

$$\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty,$$

condition (1) is valid for all sufficiently large r. By Lemma 2.4, (2) is true for all sufficiently large r. Thus γ_d exists.

For distinct $a_1, \ldots, a_q \in \mathcal{R}_d$, where $d \geq 1$ and $q \geq 3$, we set

$$X(a_1, \ldots, a_q) = \{z \in \mathbb{C} \setminus \{0, 1\}\ ;\ a_i(z) \neq a_j(z) \text{ for } i \neq j\}.$$

Then $X(a_1, \ldots, a_q)$ is a p-punctured sphere with

$$3 \leq p \leq 2d \times q(q-1)/2 + 3.$$

Namely, since

$$X(a_1, \ldots, a_q) \subset \overline{\mathbb{C}} \setminus 0, 1, \infty,$$

we have $3 \leq p$. On the other hand, we have

$$\binom{q}{2} = \frac{q!}{(q-2)!2!} = \frac{q(q-1)}{2}$$

pairs (a_i, a_j). For each pair the equation

$$a_i(z) = a_j(z)$$

has atmost $2d$ solutions, since $a_i, a_j \in \mathcal{R}_d$. Since none of the a_j needs to be the constant function $0, 1, \infty$, we have

$$p \leq 2d \times \frac{q(q-1)}{2} + 3.$$

Since $X(a_1, \ldots, a_q)$ has at least 3 holes, $X(a_1, \ldots, a_q)$ is a hyperbolic Riemann surface, that is, the universal cover of $X(a_1, \ldots, a_q)$ is the unit disc (or the upper half plane). On the contrary,

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\} = \overline{\mathbb{C}} \setminus \{0, \infty\}$$

is conformal (analytic bijection) and therefore the universal cover of $\mathbb{C} \setminus \{0\}$ is $\mathbb{C}$. We say that $\mathbb{C} \setminus \{0\}$ is a parabolic Riemann surface.

For the hyperbolic area of $X(a_1,\dots,a_q)$, we have (cf. [12,p. 233])

$$A_{\text{hyp}}(X(a_1,\dots,a_q)) = \frac{\pi}{2}(p-2) \le 2dq^2. \tag{6.3}$$

Here and what follows, we always normalize the hyperbolic metrics so that its curvature is equal to -4. Thus the hyperbolic metric on the unit disc is $|dz|/(1-|z|^2)$. For a curve γ on $X(a_1,\dots,a_q)$, we denote by $\ell_{X(a_1,\dots,a_q)}$ its hyperbolic length.

Let $\Omega \subset \mathbb{C}$ be an open set. We set

$$T(r,f,\Omega) = \frac{1}{\pi}\int_1^r A(t,f,\Omega)\frac{dt}{t},$$

where

$$A(t,f,\Omega) = \int_{D(t)\cap\Omega} f^*\omega_{\overline{\mathbb{C}}} = \int_{D(t)\cap\Omega} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\, dm(z).$$

(Hence $T(r,f,\Omega) = T(r,f\chi_\Omega)$, where χ_Ω is the characteristic function of Ω.) Let $a(z)$ be a meromorphic function on the plane which is distinct from $f(z)$. Let $f = g/h$ and $a = b/c$ be reduced representations. We put

$$\overline{N}(r,a,f,\Omega) = \int_1^r \overline{n}(t,a,f,\Omega)\frac{dt}{t},$$

where $\overline{n}(t,a,f,\Omega)$ is the number of solutions of $gc - hb = 0$ on $D(t)\cap\Omega$ ignoring multiplicity.

Proposition 6.3. *eich Let f be a transcendental meromorphic function in the complex plane and let $a_1,\dots,a_q \in \mathcal{R}_d$ be distinct with $a_q \equiv \infty$, where $d \ge 1$ and $q \ge 3$. Let $xinX(a_1,\dots,a_q)$ be a point and let $\Omega \subset X(a_1,\dots,a_q)$ be a neighbourhood of x which is a topological disc or an annulus with $\ell_X(a_1,\dots,a_q)(\partial\Omega) < 1/(2^{25}q)$. Let $0 < m < 2^{-3}$, and let $\Omega* \Subset \Omega$ be a relatively compact domain such that each connected component of $\Omega \setminus \overline{\Omega^*}$ is an annulus of modulus greater than or equal to m. For each $1 \le i \le q-2$, we take $i^\Diamond \in \{i+1,\dots,q-1\}$ such that*

$$|a_i^\Diamond(x) - a_i(x)| \le |a_j(x) - a_i(x)|, \quad j \in \{i+1,\dots,q-1\}\,. \tag{6.4}$$

Then we have

$$\begin{aligned}\sum_{i=1}^{q-2} T\left(r, \frac{f-a_i}{a_i^\Diamond - a_i}, \Omega^*\right) &\le \sum_{i=1}^{q} \overline{N}(r,f,a_i,\Omega) \\ &\quad + 2^{70}\frac{d\,q^9}{m^2} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned} \tag{6.5}$$

for all $r \in \gamma_d$.

CHAPTER 7

Derivation of Theorem 4.1 from Proposition 4.3

In this section, we shall derive Theorem 6.1 from Proposition 6.3. We first introduce a smooth (1,1)-form on the plane, which plays important role in the derivation. Let $a_1, \ldots, a_k \in \mathcal{R}_d \setminus \{infty\}$ be distinct rational functions, which are not identically equal to ∞. We define a non-negative, smooth (1,1)-form $\kappa(f, a_1, \ldots, a_k)$ by

$$\kappa(f, a_1, \ldots, a_k) = dd^c \log \left(\frac{1}{|f(z) - a_1(z)|^2} + \ldots + \frac{1}{|f(z) - a_k(z)|^2} \right)$$

outside the singular set which consists of zeros and common poles of $f - a_1, \ldots, f - a_k$. We remark that $\kappa(f, a_1, \ldots, a_k)$ extends to the whole plane as a smooth (1,1)-form. To see this, we take a meromorphic function h and entire functions $g_1, \ldots, g_k$ without common zeros such that

$$\frac{1}{f - a_1} = hg_1, \ldots, \frac{1}{f - a_k} = hg_k.$$

Then we have

$$\begin{aligned} \kappa(f, a_1, \ldots, a_k &= dd^c \log(|g_1(z)|^2 + \ldots + |g_k(z)|^2) + dd^c \log |h(z)|^2 \\ &= dd^c \log(|g_1(z)|^2 + \ldots + |g_k(z)|^2) \end{aligned}$$

outside the zeros and poles of h, since $\log |h(z)|^2$ is a harmonic function. Now the function $\log(|g_1(z)|^2 + \ldots + |g_k(z)|^2)$ is a C^∞ subharmonic function. Hence $dd^c \log(|g_1(z)|^2 + \ldots + |g_k(z)|^2)$ is a non-negative, smooth (1,1)-form on the whole plane, which proves our claim. For an open set $U \subset \mathbb{C}$, we set

$$T(r, \kappa(f, a_1, \ldots, a_k), U) = \int_1^r \left(\int_{D(t) \cap U} \kappa(f, a_1, \ldots, a_k) \right) \frac{dt}{t}.$$

The derivation of Theorem 6.1 from Proposition 6.3 consists of three steps. The first step is to derive Proposition 5.1 from Proposition 6.3. The issue is to show that

$$\sum_{i=1}^{q-2} T\left(r, \frac{f - a_i}{a_i^\diamondsuit - a_i}, \Omega(t)\right)$$

is comparable with $T(r, \kappa(f, a_1, \ldots, a_k), \Omega(t))$ modulo a small error, where $\Omega^* \subset \Omega(t) \subset \Omega$ is defined by (5.2) (cf. Lemma 5.2). A non-integrated version of this estimate is first proved (cf. Lemma 7.4), where the error term depends on the length of $\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial(\Omega(t) \cap$

$D(t)$)), then we apply length-area method to show this length is relatively small (cf. Lemma 7.5).

In the second step, we globalize Proposition 7.1 to show Proposition 7.7, which works on $D(r)$ which Proposition 7.1 works on Ω. The derivation is based on thick-thin decomposition of the punctured sphere $X(a_1, \dots, a_q)$. On the thin parts of $X(a_1, \dots, a_q)$, which consist of annuli or punctured discs with short boundaries, we may apply Proposition 7.1. On the thick parts of $X(a_1, \dots, a_q)$, we apply Proposition 7.1 over all embedded hyperbolic discs with a fixed small hyperbolic radius and average the resulting estimates. Summing these estimates for the thin parts and the thick parts, we establish Proposition 7.7.

In the final step, we estimate $T(r, \kappa(f, a_1, \dots, a_k))$ from below by

$$\int_0^{2\pi} \max_{1\le i\le q} \log \frac{1}{[f(re^{i\theta}), a_i(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{i=1}^{q} N(r, f, a_i) - 2T(r, f)$$

with a small error (cf. Lemma 7.10). This and Proposition 7.7 yield Theorem 6.1.

7.1. The first step

We derive the following proposition from Proposition 6.3.

PROPOSITION 7.1. *Let $f, a_1, \dots, a_q$ be the same as in Proposition 6.3. Let $\Omega \subset X(a_1, \dots, a_q)$ be a topological disc or an annulus with $\ell_X(a_1, \dots, a_q)(\partial\Omega) < 1/(2^{25}q)$. Let $0 < m < 2^{-3}$ be given, and let $\Omega^* \Subset \Omega$ be a relatively compact domain such that each connected component of $\Omega\setminus\overline{\Omega^*}$ is an annulus of modulus greater than or equal to m. Then we have*

$$T(r, \kappa(f, a_1, \dots, a_{q-1}, \Omega^*) \le \sum_{i=1}^{q} \overline{N}(r, f, a_i, \Omega) + 2^{73}\frac{d\, q^9}{m^2} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \tag{7.1}$$

for all $r > \gamma_d$.

To derive this proposition from Proposition 6.3, the most important task is to compare the left-hand sides of (6.5) and (7.1). This is contained in Lemma 7.2.

Let $\{A_i\}_{i=1}^{k}$ be the set of connected components of $\Omega \setminus \overline{\Omega^*}$. Here $k = 1$ if Ω is a topological disk and $k = 2$ if Ω is an annulus. Let μ_i be the modulus of the annulus A_i. Then $\mu_i \ge m$. Let

$$h_i : \left\{1 < |z| < e^{2\pi\mu_i}\right\} \to A_i$$

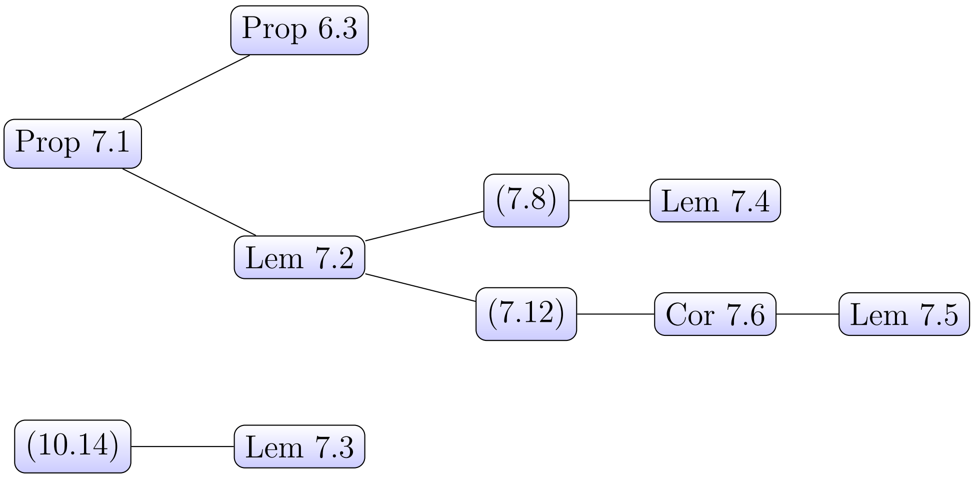


FIGURE 7.1. The first step: Proposition 7.1 is derived from Proposition 6.3 with the aid of Lemma 7.2. Lemma 7.3 is not used in this point, but later in the paper.

be a standard conformal map with $h_i(|z|=1) \subset \partial\Omega^*$. For $0<t<m$, we set

$$\Omega(t) = \overline{\Omega^*} \cup \bigcup ih_i(\{1<|z|<e^{2\pi t}\}). \tag{7.2}$$

Then $\Omega(t)$ is a domain with $\Omega^* \Subset \Omega(t) \Subset \Omega$. For $r>0$, we set $\Omega(r,t) = \Omega(t) \cap D(r)$.

Now we claim the key estimate in our derivation.

LEMMA 7.2. *Let $x \in X(a_1,\dots,a_q)$ and $\Omega^* \Subset \Omega$ be the same as in Proposition 6.3. For each $1 \le i \le q-2$, we take $i^\bullet \in \{i+1,\dots,q-1\}$ such that*

$$\frac{3}{4}|a_i^\bullet(x) - a_i(x)| \le |a_j(x) - a_i(x)| \tag{7.3}$$

for all $j \in \{i+1,\dots,q-1\}$. Then we have

$$\begin{aligned}&\int_0^m \left| \sum_{i=1}^{q-2} T\left(r, \frac{f-a_i}{a_i^\bullet - a_i}, \Omega(t)\right) - T(r, \kappa(f,a_1,\dots,a_q), \Omega(t)))\right| dt \\ &\le 2^{15}q^5 T\left(1+\frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned} \tag{7.4}$$

for $r>\gamma_d$.

This lemma is stronger than we need in our derivation, since the assumption for $i^\bullet$ is weaker than that of $i^\diamondsuit$. We shall apply Lemma 7.2 in Section 8 in this stronger form.

The proof of Lemma 7.2 is rather lengthy. We first remark that by the definition of $i^\bullet$, we have

$$|\operatorname{cr}(a_j(x), a_i(x), a_i^\bullet(x), a_q(X))| \geq \frac{3}{4}$$

for all $j \in \{i+1, \dots, q\}$. The next lemma immediately implies

$$|\operatorname{cr}(a_j(z), a_i(z), a_i^\bullet(z), a_q(z))| \geq \frac{1}{2}$$

for all $z \in \Omega$ and $j \in \{i+1, \dots, q\}$.

LEMMA 7.3. *Let $\Omega \subset X(a_1, \dots, a_q)$ be a topological disk or an annulus with $\ell_X(a_1, \dots, a_q)(\partial\Omega) < 2^{-25}$. Then, for $z, w \in \Omega$, we have*

$$[\operatorname{cr}(a_i(z), a_j(z), a_k(z), a_l(z)), \operatorname{cr}(a_i(w), a_j(w), a_k(w), a_l(w))] < 2^{-22},$$

where i, j, k and l are distinct elements in $\ell 1, \dots, q$.

PROOF. Let $\varphi : X \to \overline{\mathbb{C}}$ be a map defined by

$$\varphi(z) = \operatorname{cr}(a_i(z), a_j(z), a_k(z), a_l(z)),$$

where $X = X(a_1, \dots, a_q)$. We remark that φ omits $0, 1, \infty$, since a_i, a_j, a_k, a_l are distinct. We set $X_\Omega = \widetilde{X} / \operatorname{Im}(\pi_1(\Omega) \to \pi_1(X))$, where $\widetilde{X}$ denotes the universal covering of X. Namely, $X_\Omega \to X$ is the covering space corresponding to $\tau(\pi_1(\Omega)) \subset \pi_1(X)$, where $\tau : \pi_1(\Omega) \to \pi_1(X)$ is the induced group homomorphism (cf. [28, p.71]).

Then $\Omega \subset X_\Omega$. Note that X_Ω is an annulus when $\Omega \subset X$ is an essential annulus, otherwise $X_\Omega = \widetilde{X}$. We denote by $\psi : X_\Omega \to \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ the composition of the covering map $X_\Omega \to X$ and $\varphi : X \to \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. We consider two cases, whether lift $b : X_\Omega \to \mathbb{D}$ of ψ to the universal cover $\mathbb{D} \to \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ exists or not.

If the lift $b : X_\Omega \to \mathbb{D}$ exists, then we have $d_{\mathbb{D}}(b(z), b(w)) < 2^{-26}$, where $d_{\mathbb{D}}$ is the hyperbolic distance function on $\mathbb{D}$. Hence, we have

$$d_{\overline{\mathbb{C}} \setminus \{0,1,\infty\}}(\varphi(z), \varphi(w)) < 2^{-26},$$

where $d_{\overline{\mathbb{C}} \setminus \{0,1,\infty\}}$ is the hyperbolic distance function on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. By [6,p.267], we have

$$d_{\text{spherical}}(X, y) < 5 d_{\overline{\mathbb{C}} \setminus \{0,1,\infty\}}(x, y) \tag{7.5}$$

for $x, y \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, where $d_{\text{spherical}}$ is the spherical distance function on $\overline{\mathbb{C}}$ with respect to the line element $|dz|/(1 + |z|^2)$. Hence, we obtain our estimate.

We next consider the case when the lift $b : X_\Omega \to \mathbb{D}$ does not exist. In this case, X_Ω is an annulus. For each $\xi \in \Omega$, there exists a loop $\gamma \subset X_\Omega$ passing through ξ such that γ is homotopically non-trivial and $\ell_X(\gamma) < 2^{-25}$. We remark that

(i) the image $\psi(\gamma)$ does not lift to the covering $\mathbb{D} \to \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$,

(ii) $\ell_{\overline{\mathbb{C}}}(\psi(\gamma)) < 2^{-22}$, by (7.5), where $\ell_{\overline{\mathbb{C}}}$ denotes the length function with respect to the spherical line element $|dz|/(1+|z|^2)$.

Hence, we have $d_{\text{spherical}}(\psi(\xi), \{0, 1, \infty\}) < 2^{-23}$ for all $\xi \in \Omega$. Hence $\varphi(\Omega)$ is contained in the 2^{-23}-neighbourhood of one of $0, 1$ and ∞. We establish our estimate. □

Next we prove the following non-integrated version of (7.4).

LEMMA 7.4. [**19**, Lemma 5.4] *We have*

$$\left| \int_{\Omega(r,t)} \kappa(f, a_1, \dots, a_{q-1}) - \sum_{i=1}^{q-2} \frac{1}{\pi} \int_{\Omega(r,t)} \mathrm{cr}(f, a_i, a_i^\bullet, a_q)^* \omega_{\overline{\mathbb{C}}} \right|$$
$$\leq 2^{10} q^2 \sum_{\alpha,\beta,\gamma} \ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial\Omega(r,t)$$

Here the summation is taken over all distinct triples (α, β, γ) in the set $\{1, 2, \dots, q\}$.

Proof. To prove the lemma, it is enough to show

$$\left| \int_{\Omega(r,t)} \kappa(f, a_1, \dots, a_{q-1}) - \int_{\Omega(r,t)} \kappa(f, a_{i+1}, \dots, a_{q-1}) - \frac{1}{\pi} \int_{\Omega(r,t)} \mathrm{cr}(f, a_i, a_i^\bullet, a_q)^* \omega_{\overline{\mathbb{C}}} \right|$$
$$\leq 2^{10} q \sum_{\alpha,\beta,\gamma} \ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma))(\partial\Omega(r,t)).$$

Namely, the idea is that if

$$|K_1 - K_2 - x| < q, \quad |K_2 - K_3 - x| < q,$$

then

$$|K_1 - 2x| = |K_1 - K_2 - x + K_2 - K_3 - x| < 2q.$$

Here, we remark $\kappa(f, a_{q-1}) = 0$.

We have outside the singular set

$$dd^c \log \left(\sum_{j=i}^{q-1} \left| \frac{1}{f - a_j} \right|^2 \right) = \kappa(f, a_i, \dots, a_{q-1}),$$

and

$$dd^c \log \left(\sum_{j=i+1}^{q-1} \left| \frac{1}{f - a_j} \right|^2 \right) = \kappa(f, a_{i+1}, \dots, a_{q-1}),$$

and

$$dd^c \log \left(1 + \left| \frac{a_i^\bullet - a_i}{f - a_i} \right|^2 \right) = \frac{1}{\pi} \mathrm{cr}(f, a_i, a_i^\bullet, a_q)^* \omega_{\overline{\mathbb{C}}}.$$

Hence denoting

$$G = \frac{\sum_{i=1}^{q-1} |1/(f - a_j)|^2}{(1 + |(a_i^\bullet - a_i)/(f - a_i)|^2)(\sum_{i=1}^{q-1} |1/(f - a_j)|^2}$$

we have outside the singular set

$$dd^c \log G = \kappa(f, a_i, \dots, a_{q-1}) - \kappa(f, a_{i+1}, \dots, a_{q-1}) - \frac{1}{\pi}\,\mathrm{cr}(f, a_i, a_i^\bullet, a_q)^* \omega_{\overline{\mathbb{C}}}.$$

Set

$$G_1 = \frac{1}{1 + |(a_i^\bullet - a_i)/(f - a_i)|^2} = \frac{1}{1 + B/A_i}$$

and

$$G_2 = \frac{1}{(|f - a_i|^2 + |a_i^\bullet - a_i|^2)/(|f - a_i^\bullet|^2)} = \frac{1}{(A_i + B)/A_i^\bullet}$$

and

$$G_3 = \frac{1}{1 + \sum_{j=1, j\neq i^\bullet}^{q-1} |(f - a_i^\bullet)/(f - a_j)|^2} = \frac{1}{1 + \sum A_i^\bullet/A_j}.$$

Now

$$\begin{aligned} G_2 G_3 &= \frac{1}{(A_i + B)/A_i^\bullet} \frac{1}{1 + \sum A_i^\bullet/A_j} \\ &= \frac{1}{(A_i + B)} \frac{1}{1/A_i^\bullet + \sum 1/A_j} = \frac{1}{(A_i + B)\sum 1/A_j}. \end{aligned} \tag{7.6}$$

Then

$$\begin{aligned} G_1 + G_2 G_3 &= \frac{A_i}{A_i + B} + \frac{1}{(A_i + B)\sum 1/A_j} \\ &= \frac{A_i(\sum 1/A_j) + 1}{(A_i + B)\sum 1/A_j} \\ &= \frac{\sum A_i/A_j}{(A_i + B)\sum 1/A_j} \\ &= \frac{\sum 1/A_j}{(1 + B/A_i)\sum 1/A_j} = G \end{aligned}$$

Since $G_1 \le 1$, $G_2 \le 2$, $G_3 \le 1$, we have

$$G \le 3.$$

Claim[**19**, page 731]. If $z \in \Omega$, then

$$G(z) \ge \frac{1}{2^6 q}.$$

Let $z \in \Omega$ be fixed. For brevity, we denote

$$\mathrm{CR}(f, i, i^\bullet, q) = \mathrm{cr}(f(z), a_i(z), a_i^\bullet(z), a_q(z))$$

Proof of Claim. We consider two cases.

Case 1: $|\,\mathrm{CR}(f, i, i^\bullet, q)| \ge \frac{1}{4}$. Note that $G_1(z) = \langle 0, \mathrm{CR}(f, i, i^\bullet, q)\rangle^2$. Thus we have $G \ge G_1 \ge \frac{1}{17}$.

Case 2. $|\operatorname{CR}(f,i,i^\bullet,q)| \le \frac{1}{4}$.
First,

$$|\operatorname{CR}(f,i,q,i^\bullet)| = \left|\frac{\operatorname{CR}(f,i,i^\bullet,q)}{\operatorname{CR}(f,i,i^\bullet,q)-1}\right| \le \frac{-1/4}{1/4-1} = \frac{1}{3}.$$

Second,

$$|\operatorname{CR}(f,q,i,i^\bullet)| = \left|\frac{1}{1-\operatorname{CR}(f,i,i^\bullet,q)}\right| \le \frac{1}{1-1/4} = \frac{4}{3}.$$

Hence,

$$G_2(z) = \frac{1}{|\operatorname{CR}(f,i,q,i^\bullet)|^2 + |\operatorname{CR}(f,q,i,i^\bullet)|^2} \ge \frac{1}{(1/3)^2+(1/4)^2} = \frac{9}{17}.$$

By (5.4), we have

$$|\operatorname{CR}(s,i,q,i^\bullet)| > \frac{1}{2}, \quad s = i+1,\dots,q-1.$$

Thus, we have for $s = i+1,\dots,q-1$ and $s \ne i^{i^\bullet}$,

$$|\operatorname{CR}(f,i^\bullet,q,s)| = \left|\frac{\operatorname{CR}(f,i,i^\bullet,q)-1}{\operatorname{CR}(f,i,i^\bullet,q)-\operatorname{CR}(s,i,i^\bullet,q)}\right| \le \frac{1+1/4}{1/2-1/4} = \frac{5/4}{1/4} = 5.$$

Hence, the sum

$$\sum_{s=i+1, s\ne i^{i^\bullet}}^{q-1} |\operatorname{CR}(f,i^\bullet,q,s)|^2$$

contains at most q terms, which are all ≤ 5.
Hence, we obtain

$$G_3(z) = \frac{1}{1+\sum_{s=i+1,s\ne i^{i^\bullet}}^{q-1} |\operatorname{CR}(f,i^\bullet,q,s)|^2} \ge \frac{1}{1+25q}.$$

Thus, we have

$$G \ge G_2G_3 \ge \frac{9}{17}\cdot\frac{1}{1+25q} \ge \frac{1}{2+50q}.$$

On combining these two cases, we obtain ($q \ge 3$)

$$G \ge \min\left\{\frac{1}{17}, \frac{1}{2+50q}\right\} > \frac{1}{2^6 q}.$$

Now $\log G$ is a smooth function on Ω. Hence, we have

$$\int_{\Omega(r,t)} dd^c \log G = \int_{\partial\Omega(r,t)} d^c \log G = \int_{\partial\Omega(r,t)} \frac{d^c G}{G}.$$

Thus by the claim above, we have

$$\left|\int_{\Omega(r,t)} dd^c \log G\right| \le 2^6 q \int_{\partial\Omega(r,t)} |d^c G|.$$

Hence, by (5.7), we have

$$\left|\int_{\Omega(r,t)}\left(\kappa(f,a_i,\dots,a_{q-1})-\kappa(f,a_{i+1},\dots,a_{q-1})-\frac{1}{pi}\,\mathrm{cr}(f,a_i,a_{i^\bullet},a_q)^*\omega_{\overline{\mathbb{C}}}\right)\right| \le 2^q\int_{\partial\Omega(r,t)}|d^cG|.$$

We will estimate $|d^cG_1|$, $|d^cG_2|$ and $|d^cG_3|$. In each case, we will use the fact

$$|d^c|z|^2 \le 2|z||dz|$$

which refers to the fact that

$$\left|\frac{d}{dt}|\varphi(t)|^2\right| = \left|2\,\mathrm{Re}\,\overline{\varphi(t)}\varphi'(t)\right| \le 2|\varphi(t)||\varphi'(t)|$$

and the inequality

$$\frac{2x}{(1+x^2)^2} \le \frac{2x+(x-1)^2}{(1+x^2)^2} = \frac{1}{1+x^2}.$$

First, we have

$$|d^cG_1| \le \frac{2|(a_i^\bullet-a_i)/(f-a_i)||((a_i^\bullet-a_i)/(f-a_i))'|}{(1+|(a_i^\bullet-a_i)/(f-a_i)|^2)^2}|dz|,$$

which gives

$$|d^cG_1| \le \frac{|((a_i^\bullet-a_i)/(f-a_i))'|}{1+|((a_i^\bullet-a_i)/(f-a_i))|^2}|dz|,$$

which yields

$$\int_{\partial\Omega(r,t)}|d^cG_1| \le \ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f,a_i,a_i^\bullet,a_q)(\partial\Omega(r,t)).$$

For $w\in\mathbb{C}$, we have

$$|w|^2+|1-w|^2 \ge |w|^2+(1-|w|^2) \ge \frac{1+|w|^2}{3},$$

since

$$h(x)=3(x^2+(1-x)^2-(1+x^2)/3)=5x^2+2-6x, \quad h'(x)=10x-6=0,$$

satisfies

$$h\left(\frac{3}{5}\right)=\frac{1}{5}>0.$$

Let

$$w=\frac{f-a_i}{f-a_i^\bullet}, \quad 1-w=\frac{a_i-a_i^\bullet}{f-a_i^\bullet}$$

so that

$$G_2=\frac{1}{|w|^2+|1-w|^2}.$$

Now

$$|d^cG_2| \leq \frac{2|ww'|+2|(1-w)(1-w)'|}{|w|^2+|1-w|^2} \leq 9\frac{2|ww'|}{(1+|w|^2)^2}+9\frac{2|(1-w)(1-w)'|}{(1+|1-w|^2)^2}$$
$$\leq \frac{9|w'|}{1+|w|^2}+\frac{9|(1-w)'|}{1+|1-w|^2}.$$

Hence,

$$\int_{\partial\Omega(r,t)} |d^cG_1| \leq 9\ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f,a_i,a_q,a_i^\bullet)(\partial\Omega(r,t))+9\ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f,a_q,a_i,a_i^\bullet)(\partial\Omega(r,t)).$$

Denote

$$G_3 = \frac{1}{1+\sum_{j=i+1,j\neq i^\bullet}^{q-1} |(f-a_i^\bullet)/(f-a_j)|^2} = \frac{1}{1+\sum|\alpha_j|^2}$$

Now we have

$$|d^cG_3| \leq \frac{\sum 2|\alpha_j||\alpha_j'|}{(1+\sum|\alpha_j|^2)^2}|dz| \leq \sum \frac{|\alpha_j'|}{1+|\alpha_j|^2}|dz|.$$

Hence

$$\int_{\partial\Omega(r,t)} |d^cG_3| \leq \sum_{j=i+1,j\neq i^\bullet}^{q-1} \ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f,a_i^\bullet,a_q,a_j)(\partial\Omega(r,t))$$

Hence, we have

$$\begin{aligned}\int_{\partial\Omega(r,t)} |d^cG| &= \int_{\partial\Omega(r,t)} |d^cG_1+G_2d^cG_3+G_3d^cG_2| \\ &\leq 9\sum_{\alpha,\beta,\gamma)} \ell_{\overline{\mathbb{C}}}(\mathrm{cr}(f,a_\alpha,a_\beta,a_\gamma)(\partial\Omega(r,t)).\end{aligned} \tag{7.7}$$

This proves (5.6). We conclude the proof of Lemma 5.4.

Now we integrate both sides of the estimate of 7.4 to obtain

(7.8)

$$\int_0^m \left|\sum_{i=1}^{q-2} T\left(r,\frac{f-a_i}{a_i^\bullet-a_i},\Omega(t)\right) - T(r,\kappa(f,a_1,\dots,a_{q-1}),\Omega(t))\right| dt$$
$$\leq 2^{10}q^2\sum_{\alpha,\beta,\gamma}\int_0^m\int_1^r \ell_{\mathbb{C}}(\mathrm{cr}(f,a_\alpha,a_\beta,a_\gamma)(\partial\Omega(u,t)))\frac{du}{u}\,dt.$$

To estimate the right-hand side, we need the following.

LEMMA 7.5. *Let $\rho(z)|dz|$ be a conformal metric on Ω. Set*

$$A(r,t) = \int_{\Omega(r,t)} \rho^2(z)|dz|^2,$$
$$\ell(r,t) = \int_{\partial\Omega(r,t)} \rho(z)|dz|.$$

Let $\Lambda, \widetilde{\Lambda} : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ *be functions with*

$$\Lambda(r) \geq \max\left\{\int_1^r A(u,m)\frac{du}{u}, \log r\right\},$$
$$\widetilde{\Lambda}(r) \geq \max\left\{\Lambda(r), \Lambda\left(r + \frac{1}{\Lambda(r)}\right)\right\}.$$

Then we have

$$\int_0^m \int_1^r \frac{\ell(u,t)}{u}\, du\, dt \leq 2^2 \widetilde{\Lambda}(r)^{3/4} (\log r)^{1/4} \tag{7.9}$$

for $r > e$.

Proof. Set

$$\gamma_1(r,t) = \partial\Omega(r,t) \cap \partial\mathbb{C}(r), \quad \gamma_2(r,t) = \partial\Omega(r,t) \setminus \gamma_1(r,t),$$
$$\ell(r,t) = \int_{\gamma_1(r,t)} \rho(z)|dz|, \quad \ell_2(r,t) = \int_{\gamma_2(r,t)} \rho(z)|dz|.$$

Using the Schwarz inequality, we have

$$\ell_1(r,t)^2 \leq 2\pi r \frac{d}{dr} A(r,t).$$

We define $r_1 \in [1,r]$ according to three cases:

(i) if $A(1,t) > 1$, then $r_1 = 1$,
(ii) if $A(r,t) < 1$, then $r_1 = r$,
(iii) otherwise, we may take $r_1 \in [1,r]$ such that $A(r_1,t) = 1$.

Then we have

$$\int_1^r \ell_1(u,t)\frac{du}{u} = \int_1^{r_1} \ell_1(u,t)\frac{du}{u} + \int_{r_1}^r \ell_1(u,t)\frac{du}{u}$$
$$\leq \sqrt{2\pi}\int_1^{r_1} \sqrt{u\frac{dA}{du}(u,t)} fracduu + \sqrt{2\pi}\int_1^{r_1} \sqrt{u\frac{dA}{du}(u,t)}\frac{du}{u}.$$

We have

(7.10)

$$\int_1^{r_1} \sqrt{u\frac{dA}{du}(u,t)}\frac{du}{u} \leq \left(\int_1^{r_1} \frac{du}{u}\right)^{1/2} \left(\int_1^{r_1} \frac{d}{du}A(u,t)\, du\right)^{1/2}$$
$$\leq (\log r)^{1/2} \leq \log r,$$
$$\int_{r_1}^r \sqrt{u\frac{dA}{du}(u,t)}\frac{du}{u} = \int_{r_1}^r \sqrt{\frac{(dA/du)(u,t)}{A(u,t)}}\sqrt{\frac{A(u,t)}{u}}\, du$$
$$\leq \left(\int_{r_1}^r \frac{(dA/du)(u,t)}{A(u,t)}\, du\right)^{1/2} \left(\int_{r_1}^r \frac{A(u,t)}{u}\, du\right)^{1/2}$$
$$\leq (\log^+ A(r,t))^{1/2} \left(\int_1^r \frac{A(u,t)}{u}\, du\right)^{1/2}.$$

Hence, we have

$$\int_1^r \ell_1(u,t)\frac{du}{u} \le \sqrt{2\pi}(\log^+ A(r,t))^{1/2}\left(\int_1^r \frac{A(u,t)}{u}\,du\right)^{1/2} + \sqrt{2\pi}\log r.$$

Let $r < R < er$. Since $A(r,t)$ is increasing, we have

$$A(r,t)\log\frac{R}{r} = A(r,t)\int_r^R \frac{du}{u} \le \int_1^R A(u,t)\frac{du}{u}.$$

Hence, using $\log x \le 2\sqrt{x}$, we obtain

$$\begin{aligned}\log A(r,t) &\le -\log\log\frac{R}{r} + \log\left(\int_1^R A(u,t)\frac{du}{u}\right)\\ &\le -\log\log\frac{R}{r} + 2\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}.\end{aligned} \tag{7.11}$$

The last term is non-negative, hence

$$\log^+ A(r,t) \le -\log\log\frac{R}{r} + 2\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}.$$

Thus, by $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$, we obtain

$$\begin{aligned}\int_1^r \ell_1(u,t)\frac{du}{u} \le{}& \sqrt{2\pi}\sqrt{-\log\log\frac{R}{r}}\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}\\ &+ 2\sqrt{\pi}\left(\int_1^R A(u,t)\frac{du}{u}\right)^{3/4} + \sqrt{2\pi}\log r.\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\int_0^m\int_1^r \ell_1(u,t)\frac{du}{u}\,dt \le{}& \frac{\sqrt{2\pi}}{8}\sqrt{-\log\log\frac{R}{r}}\left(\int_1^R A(u,m)\frac{du}{u}\right)^{1/2}\\ &+ \frac{\sqrt{\pi}}{4}(\log r)^{1/4}\left(\int_1^R A(u,m)\frac{du}{u}\right)^{3/4} + \frac{\sqrt{2\pi}}{8}\log r.\end{aligned}$$

We set $R = r + 1/\Lambda(r)$. Since $(\log 2)x < \log(1+x)$ for $0 < x < 1$, we have

$$\begin{aligned}-\log\log\frac{R}{r} &= -\log\log\left(1 + \frac{1}{r\Lambda(r)}\right)\\ &< \log\frac{r\Lambda(r)}{\log 2} < \log(2r\Lambda(r)) < 2\log r + \log\Lambda(r)\\ &\le 2(\Lambda(r))^{1/2}(\log r)^{1/2} + 2\Lambda(r)^{1/2} \le 4\Lambda(r)^{1/2}(\log r)^{1/2.}\end{aligned}$$

Hence, we have

$$\int_0^m\int_1^r \ell_1(u,t)\frac{du}{u}\,dt \le \frac{3\sqrt{2\pi} + 2\sqrt{pi}}{8}\widetilde{\Lambda}(r)^{3/4}(\log r)^{1/4}.$$

Next, using the Schwarz inequality, we have

$$\ell_2(r,t)^2 \leq 2\frac{d}{dt}A(r,t).$$

We have

$$\begin{aligned}\int_0^m \ell_2(r,t)\,dt &\leq \sqrt{2}\int_0^m \sqrt{\frac{dA}{dt}(r,t)}\,dt \\ &\leq \sqrt{2}\left(\int_0^m dt\right)\left(\int_0^m \frac{d}{dt}A(r,t)\,dt\right)^{1/2} \\ &\leq \sqrt{2}A(r,m)^{1/2}.\end{aligned}$$

Since

$$\int_1^r (A(u,m))^{1/2}\frac{du}{u} \leq \sqrt{\log r}\left(\int_1^r A(u,m)\frac{du}{u}\right)^{1/2},$$

we obtain

$$\begin{aligned}\int_0^m\int_1^r \ell_2(r,t)\frac{du}{u}\,dt &\leq \sqrt{2}\sqrt{\log r}\left(\int_1^r A(u,m)\frac{du}{u}\right)^{1/2} \\ &\leq \sqrt{2}\widetilde{\Lambda}(r)^{3/4}(\log r)^{1/4}.\end{aligned}$$

Since $3\sqrt{2\pi}+2\sqrt{pi} < 3\sqrt{9}+2\sqrt{4} = 13 < 2^4$, we obtain our estimate. □

Applying Lemma 7.5 to the case $\Lambda(r) = 2\pi T(r)$ and $\widetilde{\Lambda}(r) = 2\pi T(r+1/2T(r))$ for $r > \gamma_d$, we obtain[1]

COROLLARY 7.6. *For $r > \gamma_d$, we have*

(7.12)
$$\int_0^m\int_1^r \ell_{\mathbb{C}}(\mathrm{cr}(f,a_i,a_j,a_k)(\partial\Omega(u,t))) \leq 2^5 T\left(r+\frac{1}{2T(r)}\right)^{3/4}(\log r)^{1/4}.$$

Now we obtain (5.3) by substituting (7.12) to (7.8). Thus, we have proved Lemma 5.2.

Derivation of Proposition 7.1 from Proposition 6.3 Let $\Omega^* \Subset \Omega \subset X(a_1,\dots,a_q)$ be the same as in Proposition 7.1. We take a point $x \in \Omega$ and choose $i^\diamondsuit$ for each $1 \leq i \leq q-2$ as in Proposition 6.3. The estimate (7.4) implies for $r > \gamma_d$

$$\begin{aligned}T(r,\kappa(f,a_1,\dots,a_{q-1}),\Omega^*) &\leq \sum_{i=1}^{q-2} T\left(r,\frac{f-a_i}{a_i^\bullet - a_i},\Omega(m/2)\right) \\ &\quad + 2^{72}\frac{d\,q^9}{m^2}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log)^{1/4}\end{aligned}$$

for $r > \gamma_d$. This proves Proposition 7.1. □

[1]constants: $2^2(2\pi)^{3/4} < 4\cdot 2\cdot 2^2 = 2^5$

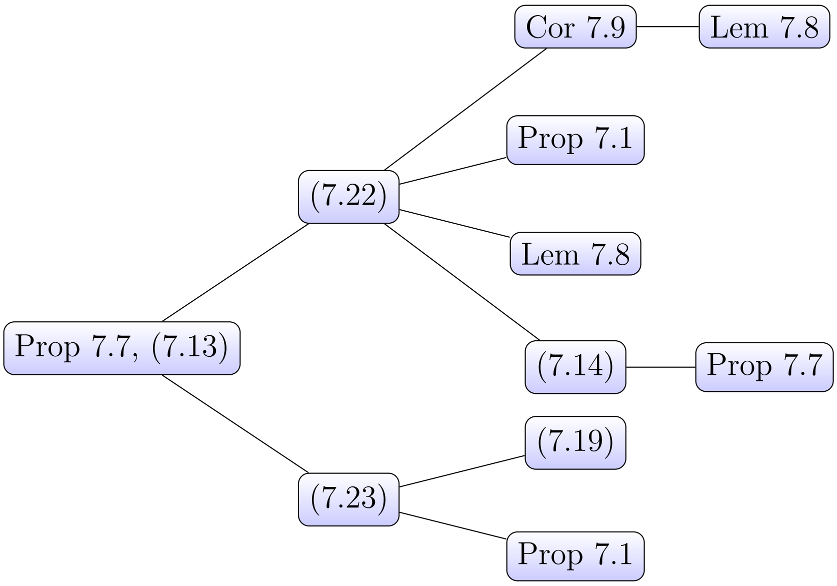


FIGURE 7.2. The second step: Proof of Proposition 7.7.

7.2. The second step

In Proposition 7.1, we treat a local value distribution of f over a topological disk or an annulus whose boundary is short. In this step, we consider global value distribution. We shall derive the following proposition from Proposition 7.1.

PROPOSITION 7.7. *Let $f, a_1, \ldots, a_q$ be the same as in Proposition 6.3. Given $\varepsilon > 0$, we have*

$$\begin{aligned}(7.13)\qquad &(1-\varepsilon)T(r, \kappa(f, a_1, \ldots, a_q)) \\ &\leq \sum_{i=1}^{q} \overline{N}(r, f, a_i) + 2^{153}\frac{d^2 q^{13}}{\varepsilon^4} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for all $r > \gamma_d$.

To derive this proposition from Proposition 7.1, we use thick-thin decomposition of the punctured sphere $X = X(a_1, \ldots, a_q)$ (cf. [7,Theorem 4.4.6]): For $\delta < \operatorname{arcsinh}(1)/2$, let $A_1, \ldots, A_k$ be the connected components of $X_{<\delta}$, where $X_{<\delta}$ denotes the subset of X with hyperbolic injectivity radius less than δ. Here the hyperbolic injectivity radius at a point $x \in X$ is the radius of the largest embedded hyperbolic ball centred at X. Then each A_i is either a horoball neighbourhood of a cusp or a collar neighbourhood of a closed geodesic of length less than 2δ. The number k satisfies the bound $k \leq 2p - 3$, where p is the number of the punctures of X. Since $p \leq dq^2$, we have

$$k < 2dq^2. \qquad (7.14)$$

LEMMA 7.8. *Let $\delta < \frac{1}{4}$ and let A be a connected component of $X_{<\delta}$. Let C be a boundary circle of A. Then*

$$\ell_X(C) \leq 4\delta. \tag{7.15}$$

PROOF. Let $\varpi : X_C \to X$ be the covering space corresponding to $\langle C \rangle \subset \pi_1(X)$. Then X_C is an annulus or a punctured disk. We identify X_C with $A(R) = \{z; 1 < |z| < R\}$ and A with

$$\left\{ z; s < |z| < \frac{R}{s} \right\},$$

where $1 < s < \sqrt{R}$. When A is a cusp neighbourhood, then $R = \infty$. The hyperbolic metric on $A(R)$ is given by

$$\frac{\pi/\log R}{2\sin(\pi \log|z|/\log R)} \frac{|dz|}{|z|}.$$

We note that this metric converges to the hyperbolic metric of the punctured disc as $R \to \infty$:

$$\frac{\pi/\log R}{2\sin(\pi \log|z|/\log R)} \frac{|dz|}{|z|} \to \frac{\pi \log|z|/\log R}{2\sin(\pi \log|z|/\log R)} \frac{|dz|}{|z|\log|z|} \to \frac{|dz|}{2|z|\log|z|}$$

as $R \to \infty$, since $\lim_{x\to 0} \sin(x)/x = 1$. For $1 < r < R$, we denote by C_r the circle $|z| = r$ in $A(R)$ and by $\eta(r)$ the hyperbolic length of C_r. Then we have

$$\eta(r) = \pi \frac{\pi/\log R}{\sin(\pi \log r/\log R)}.$$

We may take $s < t < \sqrt{R}$ such that $\eta(t) = 2\delta$. We claim that

$$\log \frac{t}{s} < \pi. \tag{7.16}$$

To show this, we take a point $a \in C_s$ and a closed essential loop *gamma* $\subset A(R)$ passing through a such that the hyperbolic length satisfies $\ell(\gamma) = 2\delta$. If γ and C_t do not intersect, then $\ell(\gamma) > 2\delta$, which is a contradiction. Hence, γ and C_t intersect. This shows

$$\mathrm{dist}(C_s, C_t) < \delta,$$

where $\mathrm{dist}(C_s, C_t)$ is the hyperbolic distance of C_s and C_t. On the other hand, we have

$$\begin{aligned} \mathrm{dist}(C_s, C_t) &= \frac{1}{2}\int_s^t \frac{\pi/\log R}{\sin(\pi \log x/\log R)} \frac{dx}{x} \\ &= \frac{1}{2\pi}\int_s^t \eta(x)\frac{dx}{x} \\ &> \frac{1}{2\pi}\eta(t)\log\frac{t}{s}. \end{aligned}$$

This shows (7.16).

Now by (7.16), we have

(7.17)
$$\sin\left(\frac{\pi\log t}{\log R}\right) \le \sin\left(\frac{\pi\log s}{\log R}\right) + \frac{\pi\log(t/s)}{\log R} \le \sin\left(\frac{\pi\log t}{\log R}\right) + \frac{\pi^2}{\log R}.$$

since

$$\sin(x+y) = \sin(x)\cos(y)+\sin(y)\cos(x) \le \sin(X)+\sin(y) \le \sin(x)+y$$

for $x, y \in \mathbb{R}$. Since $\eta(t) < \frac{1}{2}$, we have

$$\frac{\pi^2}{\log R} < \frac{1}{2}\sin\left(\frac{\pi\log t}{\log R}\right).$$

Thus,

$$\sin\left(\frac{\pi\log t}{\log R}\right) \le 2\sin\left(\frac{\pi\log s}{\log R}\right).$$

Hence, we obtain

$$\ell_X(C) = \eta(s) \le 2\eta(t) = 4\delta.$$

□

For $x \in X$, we denote by $\rho(x)$ the hyperbolic injectivity radius at X.

Corollary 7.9. *Let δ and A be the same as in Lemma 5.8. For $\delta < \delta' < \frac{1}{4}$, let A' be the connected component of $X_{<\delta'}$ such that $A \subset A'$. Let B be a connected component of $A' - \overline{A}$. Then B is an annulus whose modulus μ satisfies*

$$\mu > \frac{\delta' - \delta}{4\delta'}. \tag{7.18}$$

Proof. Let $\varpi : A(R) \to X$ be the covering as in the proof of Lemma 7.8. We identify A with $\{z; s < |z| < R/s\}$, where $1 < s < \sqrt{R}$. Then A' corresponds to

$$\left\{z;\, e^{-2\pi\mu}s < |z| < e^{2\pi\mu}R/s\right\}.$$

We may assume without loss of generality that B corresponds to

$$\left\{z;\, e^{-2\pi\mu}s < |z| < e^{2\pi\mu}s\right\}.$$

Then using the notation in the proof of Lemma 7.8, we have

$$\begin{aligned}\operatorname{dist}(C_e^{-2\pi\mu}s, C_s) &= \frac{1}{2}\int_{e^{-2\pi\mu}s}^{s} \frac{\pi/\log R}{\sin(\pi\log x/\log R)}\frac{dx}{x} \\ &= \frac{1}{2\pi}\int_{e^{-2\pi\mu}s}^{s} \eta(X)\frac{dx}{x} \\ &< m\eta\left(e^{-2\pi\mu}s\right).\end{aligned}$$

By Lemma 7.8, we have $\eta\left(e^{-2\pi\mu}s\right) < 4\delta'$. Hence, we have

$$\mu > \frac{\operatorname{dist}(C_{e^{-2\pi\mu}s}, C_s)}{4\delta'}.$$

On the other hand, we obviously have $|\rho(x) - \rho(x')| \leq \operatorname{dist}(x, x')$ for all $x, x' \in X$. Hence, we have

$$\operatorname{dist}(\partial X_{<\delta}, \partial X_{<\delta'}) \geq \delta' - \delta. \tag{7.19}$$

Hence, we obtain (7.18). □

For $\delta > 0$ and $\delta' > \delta$, we set

$$X_{\geq\,\delta} = \{x \in X;\, \rho(X) > \delta\}\,, \quad X_{[\delta,\,\delta')} = \{x \in X;\, \delta \leq \rho(x) < \delta'\}\,.$$

Derivation of Proposition 7.7 from Proposition 7.1 It is enough to consider the case $\varepsilon < 1$, for otherwise the estimate (7.13) is obvious. We take a large integer L such that

$$\frac{5}{\varepsilon} < L < \frac{8}{\varepsilon}. \tag{7.20}$$

Set $\sigma = 1/(2^{29}q)$. Since

$$X_{[\sigma + j(\sigma/L),\, \sigma + (j+1)(\sigma/L))}, \quad j = 0, 1, \ldots, L-1,$$

are disjoint, we have

$$\sum_{j=0}^{L-1}\left(T(r, \kappa, X_{[\sigma + j(\sigma/L),\, \sigma + (j+1)(\sigma/L))}) + \sum_{t=1}^{q} \overline{N}(r, f, a_t, X_{[\sigma + j(\sigma/L),\, \sigma + (j+1)(\sigma/L}\right.$$

$$T(r, \kappa) + \sum_{t=1}^{q} \overline{N}(r, f, a_t).$$

Here $\kappa = \kappa(f, a_1, \ldots, a_{q-1})$. We choose $0 \leq j \leq L-1$ which minimizes

$$T(r, \kappa, X_{[\sigma + j(\sigma/L),\, \sigma + (j+1)(\sigma/L))}) + \sum_{t=1}^{q} \overline{N}(r, f, a_t, X_{[\sigma + j(\sigma/L),\, \sigma + (j+1)(\sigma/L))})$$

and set $\tau = \sigma + j(\sigma/L)$, $\tau' = \sigma + (j+1)(\sigma/L)$. Then we have

$$\begin{aligned} &T(r, \kappa, X_{[\tau,\, \tau')}) + \sum_{t=1}^{q} \overline{N}(r, f, a_t, X_{[\tau,\, \tau')}) \\ &\frac{1}{L}\left(T(r, \kappa) + \sum_{t=1}^{q} \overline{N}(r, f, a_t)\right). \end{aligned} \tag{7.21}$$

In the following, we shall prove the following two estimates for $r > \gamma_d$:

$$\begin{aligned} T(r, \kappa, X_{<\,\tau} &\leq \sum_{t=1}^{q} \overline{N(r, f, a_t, X_{<\,\tau'})} \\ &\quad + 2^{80} d^2 q^{11} L^2 T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}, \end{aligned} \tag{7.22}$$

and

(7.23)

$$\left(1-\frac{3}{4}\right) T(r,\kappa,X_{\geq \tau'} \leq \sum_{i=1}^{q} \overline{N}(r,f,a_i) \\ + 2^{141}d^2q^1 3L^4 T\left(r+\frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}.$$

for $r > \gamma_d$. Hence by $1-5/L < (1-4/L)(1+1/L)$ and (7.20), we obtain (7.13).

Now it remains to prove (7.22) and (7.23). We first prove (7.22).

Let $A_1, \dots, A_k$ be the connected components of $X_{< \tau}$. Then each A_i is either a horoball neighbourhood of a cusp or a collar neighbourhood of a geodesic of length less than 2τ. Let A_i' be the connected component of $X_{< \tau'}$ such that $A_i \subset A_i'$. Let μ be the modulus of a connected component of $A_i' - \overline{A_i}$. Then by (7.18), we have

$$\mu \geq \frac{\tau'-\tau}{4\tau'} > \frac{1}{8L}.$$

We first assume that A_i is a collar neighbourhood of a geodesic. By (7.15), we may apply Proposition 7.1 for $\Omega = A_i'$ and $\Omega^* = A_i$ to obtain

$$T(r,\kappa,A_i) \leq \overline{N}(r,f,a_i,A_i') \\ + 2^{79}dq^9L^2T\left(r+\frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}.$$

If A_i is a horoball neighbourhood of a cusp, this estimate is still true by a limiting argument; First we take a small constant $0<\delta<\tau$, and remove $A_i \cap \overline{X_{< \delta}}$ from A_i to obtain an annulus B_i. Next, we remove a small horoball neighbourhood from A_i' to obtain an annulus B_i' so that B_i is relatively compact in B_i' and two connected components of $B_i' - \overline{B_i}$ are annuli of the same modulus μ. Then we apply Proposition 7.1 for $\Omega = B_i'$, $\Omega^* = B_i$ and finally let $\delta \to 0$.

Thus, by (7.14), we obtain (7.22).

Next we prove (7.23). For $x \in X$, we denote by $D(x)$ the hyperbolic $1/(2^{30}qL)$-ball centered at x. Then, for $x \in X_{\geq \tau}$, $D(x)$ is an embedded ball. Let $D^*(x) \subset D(x)$ be the hyperbolic ball centred at x such that the modulus of the annulus $D(x) \setminus \overline{D^*(x)}$ is equal to $1/(8L)$. For $x \in X_{\geq \tau}$, the hyperbolic areas of $D(x)$ and $D^*(x)$ are constants independent of x. We denote these constants by

$$\alpha = A_{\text{hyp}}(D(x)), \quad \alpha^* = A_{\text{hyp}}(D^*(x)).$$

For $x \in X_{\geq\, \tau}$, we apply Proposition 7.1 for $\Omega = D(x)$ and $\Omega^* = D^*(x)$ to obtain

$$\begin{aligned} T(r, \kappa, D^*(x)) \leq \sum_{i=1}^{q} \overline{N}(r, f, a_i, D(x)) \\ + 2^{79} d q^9 L^2 T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \end{aligned} \tag{7.24}$$

for $r > \gamma_d$. We set

$$Y = \left\{ z \in X;\ \mathrm{dist}(z, X_{\geq\, \tau'} \right\} < \frac{1}{2^{30} q L}. \tag{7.25}$$

We integrate both sides of (7.24) over Y with respect to the hyperbolic area of X. Then by (6.3), we obtain

$$\begin{aligned} &\int_Y T(r, \kappa, D^*(y)) dA_{\mathrm{hyp}}(y) \\ &\leq \int_Y \sum_{i=1}^{q} \overline{N}(r, f, a_i, D(y)) A_{\mathrm{hyp}}(y) \\ &+ 2^{80} d^2 q^{11} L^2 T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}. \end{aligned} \tag{7.26}$$

We note that $D^*(x)$ is contained in Y for $x \in X_{\geq\, \tau'}$. Hence, for $x \in X_{\geq\, \tau'}$, we have

$$\int_{\{y \in Y\,;\, x \in D^*(y)\}} dA_{\mathrm{hyp}}(y) = A_{\mathrm{hyp}}(D^*(x)) = \alpha^*.$$

We set $\kappa = \widetilde{\kappa}(X) dx \wedge d\overline{x}$. Then $\widetilde{\kappa}(x)$ is a non-negative, smooth function. By Fubini's theorem, we have

(7.27)
$$\begin{aligned} &\int_Y T(r, \kappa, D^*(y)) dA_{\mathrm{hyp}}(y) \\ &\int_Y \int_1^r \int_{D^*(y) \,\cap\, D(t)} \widetilde{\kappa} dx \wedge d\overline{x} \frac{dt}{t} dA_{\mathrm{hyp}}(y) \\ &= \int_1^r \int_{x \,\in\, D(t)} \left(\int_{\{y \in Y\,;\, x \in D^*(y)\}} dA_{\mathrm{hyp}}(y) \right) \widetilde{\kappa} dx \wedge d\bar{x} \frac{dt}{t} \\ &= \alpha^* T(r, \kappa, X_{\geq\, \tau'}. \end{aligned}$$

Next, by (7.19), $D(x)$ is contained in $X_{\geq\, \tau}$ for $x \in Y$. Hence, for $x \in X_{<\, \tau}$, we have

$$\int_{\{y \in Y\,;\, x \in D^*(y)\}} dA_{\mathrm{hyp}}(y) = 0.$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}
&\int_Y \sum_{i=1}^q \overline{N}(r, f, a_i, D(y)) dA_{\mathrm{hyp}}(y) \\
&= \int_Y \int_1^r \int_{D(y)\,\cap\,\mathbb{C}(t)} d\nu \frac{dt}{t} dA_{\mathrm{hyp}}(y) \\
&= \int_1^r \int_{x \in \mathbb{C}(t)} \int_{\{y\, \in\, Y\,;\; x\, \in\, D(y)\}} dA_{\mathrm{hyp}}(y)\, d\nu \frac{dt}{t} \\
&\leq \alpha \sum_{i=1}^q \overline{N}(r, f, a_i, X_{\geq\, \tau}.
\end{aligned} \tag{7.28}$$

Here, ν is a measure such that

$$\nu(A) = \sum_{i=1}^q \# \left\{ z \in A\,;\, f(z) = a_i(z) \right\}.$$

Hence, by (7.26)-(7.28), we obtain

$$\begin{aligned}
\alpha^* T(r, \kappa, X_{\geq\, \tau'} \leq \alpha \sum_{i=1}^q \overline{N}(r, f, a_i, X_{\geq\, \tau} \\
+ 2^{80} d^2 q^{11} L^2 T \left(r + \frac{1}{T(r)} \right)^{3/4} (\log r)^{1/4}.
\end{aligned}$$

Now to conclude the proof of (7.23), what we need to prove is as follows:

$$\alpha \geq \frac{1}{2^{60} q^2 L^2}, \quad \alpha^* \geq \left(1 - \frac{3}{L} \right) \alpha.$$

The first estimate follows from the fact that the area of the hyperbolic r-ball is greater than πr^2. For the second estimate, we note that $A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r)) = \pi r^2/(1-r^2)$ for $0 \leq r < 1$. Hence, for $0 \leq r < \frac{1}{2}$, we have

$$\begin{aligned}
A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(e^{-1/L} r)) &= \frac{\pi e^{-2/L} r^2}{1 - e^{-2/L} r^2} \\
&\geq e^{-3/L} \frac{\pi r^2}{1 - r^2} \\
&= e^{-3/L} A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r)).
\end{aligned}$$

Thus, we have $A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(e^{-1/L} r) \geq (1 - 3/L) A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r))$ for $0 \leq r < \frac{1}{2}$. This shows the second estimate.

7.3. The final step

We derive Theorem 6.1 from Proposition 7.7. We need the following lemma:

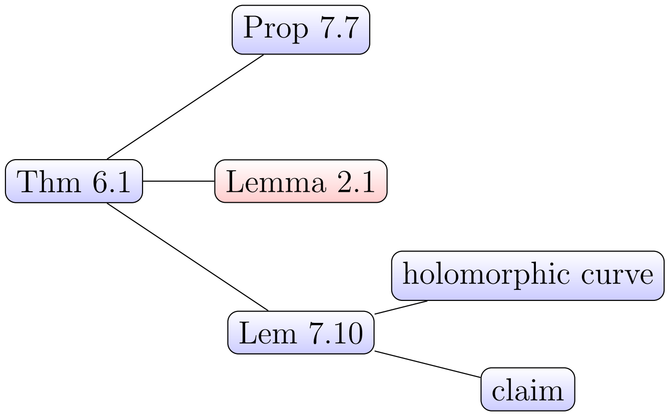


FIGURE 7.3. The final step: deriving Theorem 6.1 from Proposition 7.7.

LEMMA 7.10. *Let* $a_1, \ldots, a_q \in \mathcal{R}_d$ *be distinct with* $a_1 \equiv 0$ *and* $a_q \equiv \infty$. *Then we have*

$$\int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N(r,f,a_j) - 2T(r,f) \tag{7.29}$$
$$\le T(r, \kappa(f, a_1, \ldots, a_{q-1})) + dq^2 \log r + q(C_{f,d}+1).$$

Here, we recall the constant $C_{f,d}$ from (2.3).

Proposition 7.7 and Lemma 7.10 imply Theorem 6.1. We take a positive constant $\gamma_d' > \gamma_d$ such that the following two estimates are valid for all $r > \gamma_d'$:

$$\begin{aligned} d\log r + C_{f,d} + 1 &< T(r)^{3/4}(\log r)^{1/4} \\ 2^{180} d^2 T(r)^{3/4}(\log r)^{1/4} &< T(r)^{4/5}(\log r)^{1/5}. \end{aligned} \tag{7.30}$$

We set

$$E = \left\{ r > 1\,;\, T\left(r + \frac{1}{T(r)}\right) > 2T(r) \right\}.$$

Then by Borel's growth lemma, Lemma 2.1, E has finite linear measure. We set

$$E_{f,d} = \{r; 0 < r < \gamma_d'\} \cup E.$$

We note that $E_{f,d}$ only depends on f and d.

By Lemma 2.1 and (7.30), we have

$$\begin{aligned} &m(r,f,a_i) + N(r,f,a_i) \\ &\le T(r,f) + T(r)^{3/4}(\log r)^{1/4} \le 2T(r,f) \end{aligned} \tag{7.31}$$

for $r > \gamma_d'$. Thus, the estimate of Theorem 6.1 is obvious if $\varepsilon \ge 2q$ or $q \le 2$. In the following, we assume that $e < 2q$ and $q \ge 3$.

We first consider the special case that $a_1 = 0$ and $a_q = \infty$. By (7.29) and (7.30), we have

$$\int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N(r, f, a_j) - 2T(r, f)$$
$$\le \left(1 + \frac{\varepsilon}{2q}\right) \sum_{i=1}^{q} \overline{N}(r, f, a_i)$$
$$+ 2^{162} \frac{d^2 q^{17}}{\varepsilon^4} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}$$

for $r > \gamma'_d$. Here, we remark that $1/(1-\varepsilon/4q) < (1+\varepsilon/2q) < 2$. Hence, by (7.31), we have

$$\int_0^{2\pi} \max_{1\le j\le q} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N_1(r, f, a_j)$$
$$\le (2+\varepsilon)T(r, f) + \frac{q^{17}}{2^{17}\varepsilon^4} T(r)^{4/5} (\log r)^{1/5}$$

for all $r > 0$ outside $E_{f,d}$.

For the general case, we add two constant functions 0 and ∞ to the set $\{a_1, \dots, a_q\}$, if necessary, to reduce to the special case above. Note that in this reduction, the number q is at most replaced by $q+2$, which is smaller than $2q$.

Proof of Lemma 7.10

We need some estimates involving chordal distance.

Claim. For $w, a_1, \dots, a_k \in \mathbb{C}$, we set

$$\Lambda(w, a_1, \dots, a_k) = \frac{1}{2} \log \left(1 + \sum_{i=1}^{k} \frac{|w|^2}{|w - a_i|^2}\right) + \log \frac{1}{[w, 0]} + \log \frac{1}{[w, \infty]}.$$

Then we have:

$$\Lambda(w, a_1, \dots, a_k) \le \sum_{i=1}^{k} \log \frac{1}{[w, a_i]} + \log \frac{1}{[w, 0]} + \log \frac{1}{[w, \infty]} + k, \tag{7.32}$$

and

$$\max \left\{\log \frac{1}{[w, a_1]} + \log \frac{1}{[w, a_k]} + \log \frac{1}{[w, 0]} + \log \frac{1}{[w, \infty]}\right\} \le \Lambda(w, a_1, \dots, a_k) + 2 \tag{7.33}$$

PROOF. We first prove (7.32). We have

$$
\begin{aligned}
&\Lambda(w, a_1, \dots, a_k) \\
&\leq \max_{1\leq i\leq k}\left\{\frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a_i|^2}\right)\right\} + \log\frac{1}{[w,0]} + \log\frac{1}{[w,\infty]} + \frac{\log k}{2} \\
&\leq \max_{1\leq i\leq k}\left\{\log\frac{1}{[w,a_i]} + \frac{\log 2}{2}\right\} + \log\frac{1}{[w,0]} + \log\frac{1}{[w,\infty]} + \frac{\log k}{2} \\
&\leq \sum_{1\leq i\leq k}\log\frac{1}{[w,a_i]} + \log\frac{1}{[w,0]} + \log\frac{1}{[w,\infty]} + \frac{\log 2k}{2}.
\end{aligned}
$$

Since $\log 2k < 2k$, we obtain (7.32).

To prove (7.33), we first show that

(7.34)
$$
\log\frac{1}{[w,a]} \leq \frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a|^2}\right) + \log\frac{1}{[w,0]} + \log\frac{1}{[w,\infty]} + 2
$$

for all $w, a \in \mathbb{C}$. Indeed since $|w| \geq |a|/2$ or $|w-a| \geq |a|/2$, we have

$$
|w-a|^2 + |w|^2 \geq \frac{1}{4}|a|^2.
$$

Hence, if $|a| \geq 1$, we have

$$
\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right) \geq \frac{|a|^2}{4(1+|a|^2)} \geq \frac{1}{8}.
$$

If $|a| < 1$, we have

$$
\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right) \geq \frac{|w|^2}{2}\left(1+\frac{1}{|w|^2}\right) \geq \frac{1}{2}.
$$

Thus, we obtain

$$
\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right) \geq \frac{1}{8} > \frac{1}{e^4}.
$$

By this estimate, we have

$$
\begin{aligned}
2\log\frac{1}{[w,a]} &= \log\frac{(1+|w|^2)(1+|a|^2)}{|w-a|^2} \\
&= \log\left(1+\frac{|w|^2}{|w-a|^2}\right) + \log(1+|w|^2) + \log\left(\frac{1+|a|^2}{|w-a|^2+|w|^2}\right) \\
&\leq \log\left(1+\frac{|w|^2}{|w-a|^2}\right) + \log(1+|w|^2) + \log\left(1+\frac{1}{|w|^2}\right) + 4 \\
&= \log\left(1+\frac{|w|^2}{|w-a|^2}\right) + 2\log\frac{1}{[w,\infty]} + 2\log\frac{1}{[w,0]} + 4.
\end{aligned}
$$

This proves (7.34).

Now by (7.34), we have

$$\log\frac{1}{[w,a_i]}\leq\frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a_i|^2}\right)+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+2$$
$$\Lambda(w,a_1,\ldots,a_k)+2.$$

This proves (7.33). □

Now we prove Lemma 7.10. Define a holomorphic curve $F:\mathbb{C}\to\mathbb{P}^{q-2}$ by

$$F(z)=\left[\frac{1}{f-a_1}:\cdots:\frac{1}{f-a_{q-1}}\right].$$

Using the notation from (3.9), we have

$$T(r,\kappa(f,a_1,\ldots,a_{q-1})=T(r,F).$$

Thus, by (3.11), we have

$$T(r,\kappa(f,a_1,\ldots,a_{q-1})=N(r,F,H)+m(r,F,H)-m(1,F,H). \tag{7.35}$$

We shall estimate the right-hand side of (7.35). By the definition of the Weil function λ_H, see (3.10), we have

$$\begin{aligned}\lambda_H\circ F(z)&=\Lambda(f(z),a_2(z),\ldots,a_{q-1}(z))\\&\quad-\log\frac{1}{[f(z),0]}-\log\frac{1}{[f(z),\infty]}.\end{aligned}$$

Thus, we have

$$\begin{aligned}m(r,F,H)=&\int_0^{2\pi}\Lambda(f(re^{i\theta}),a_2(re^{i\theta}),\ldots,a_{q-1}(re^{i\theta}))\frac{d\theta}{2\pi}\\&-m(r,f,0)-m(r,f,\infty).\end{aligned}$$

Hence, by (7.32), we have

$$m(1,F,H)\leq(q-2)C_{f,d}+q-2. \tag{7.36}$$

By (7.33), we have

$$\int_0^{2\pi}\max_{1\leq j\leq q}\log\frac{1}{[f(re^{i\theta}),a(re^{i\theta})]}\frac{d\theta}{2\pi}\leq m(r,F,H)+m(r,f,0)+m(r,f,\infty)+2. \tag{7.37}$$

By (7.35)-(7.37), we obtain

$$\begin{aligned}&N(r,F,H)+\int_0^{2\pi}\max_{1\leq j\leq q}\log\frac{1}{[f(re^{i\theta}),a(re^{i\theta})]}\frac{d\theta}{2\pi}\\&-m(r,f,0)-m(r,f,\infty)\\&\leq T(r,\kappa(f,a_1,\ldots,a_{q-1})+(q-2)C_{f,d}+q.\end{aligned} \tag{7.38}$$

Next we claim

$$\sum_{i=2}^{q-1}N(r,f,a_i)\leq N(r,F,H)+dq^2\log r. \tag{7.39}$$

To show this, we take reduced representations $f = g/h$ and $a_i = b_i/c_i$, where b_i and c_i are polynomials of degree less than or equal to d. Since

$$F(z) = \left[\frac{c_1}{c_1 g - b_1 h} : \cdots : \frac{c_{q-1}}{c_{q-1} g - b_{q-1} h}\right],$$

we have

(7.40)

$$N(r, F, H) = \int_1^r \sum_{z \in \mathbb{C}(t)} \max_{2 \le i \le q-1} \left\{0, \mathrm{ord}_z \frac{c_1}{c_1 g - b_1 h} \frac{c_i g - b_i h}{c_i}\right\} \frac{dt}{t}.$$

Hence, we have

(7.41)

$$N(r, F, H) \ge \int_1^r \sum_{z \in \mathbb{C}(t)} \max_{2 \le i \le q-1} \left\{0, \mathrm{ord}_z \frac{c_i g - b_i h}{c_1 g - b_1 h}\right\} \frac{dt}{t} - \sum_{i=2}^{q-1} N(r, c_i, 0).$$

Since

$$\min\left\{\mathrm{ord}_z(c_i g - b_i h), \mathrm{ord}_z(c_j g - b_j h)\right\} \le \mathrm{ord}_z(b_i c_j - b_j c_i),$$

we have

$$\begin{aligned}\sum_{i=2}^{q-1} \mathrm{ord}_z(c_i g - b_i h) &\le \max_{2 \le i \le q-1} \left\{\mathrm{ord}_z(c_i g - b_i h)\right\} + \sum_{2 \le i < j \le q-1} \mathrm{ord}_z(b_i c_j - b_j c_i) \\ &\le \max\left\{0, \mathrm{ord}_z \frac{c_i g - b_i h}{c_1 g - b_1 h}\right\} + \sum_{1 \le i < j \le q-1} \mathrm{ord}_z(b_i c_j - b_j c_i).\end{aligned}$$

Combined with (7.41), we obtain

$$\sum_{i=2}^{q-1} N(r, f, a_i) \le N(r, F, H) + \sum_{1 \le i < j \le q-1} N(r, b_i c_j - b_j c_i, 0) + \sum_{i=2}^{q-1} N(r, c_i, 0).$$

This shows (7.39).

Now by (7.38) and (7.39) and

$$m(r, f, 0) + N(r, f, 0) + m(r, f, \infty) + N(r, f, \infty) \le 2T(r, f) + 2C_{f,d},$$

we obtain (7.29). □

CHAPTER 8

Holomorphic motion and quasiconformal perturbation

We begin the proof of Proposition 6.3. Our goal of this section is to perturb f quasiconformally and construct a quasimeromorphic function g over Ω which has appropriate properties to show Proposition 6.3. Our main tool is holomorphic motion, which we introduce below.

A *holomorphic motion* of a set $A \subset \overline{\mathbb{C}}$ over a connected complex manifold with base point (Y, y) is a mapping $\phi : Y \times A \to \overline{\mathbb{C}}$, given by $(\lambda, z) \mapsto \phi_\lambda(z) = \phi(\lambda, z)$, such that

(1) For each fixed $z \in A$, $\phi_\lambda(z)$ is a holomorphic function of λ,
(2) For each fixed $\lambda \in Y$, $\phi_\lambda(z)$ is an injective function of z,
(3) The injection is the identity at the base point, that is, $\phi_y(z) = z$.

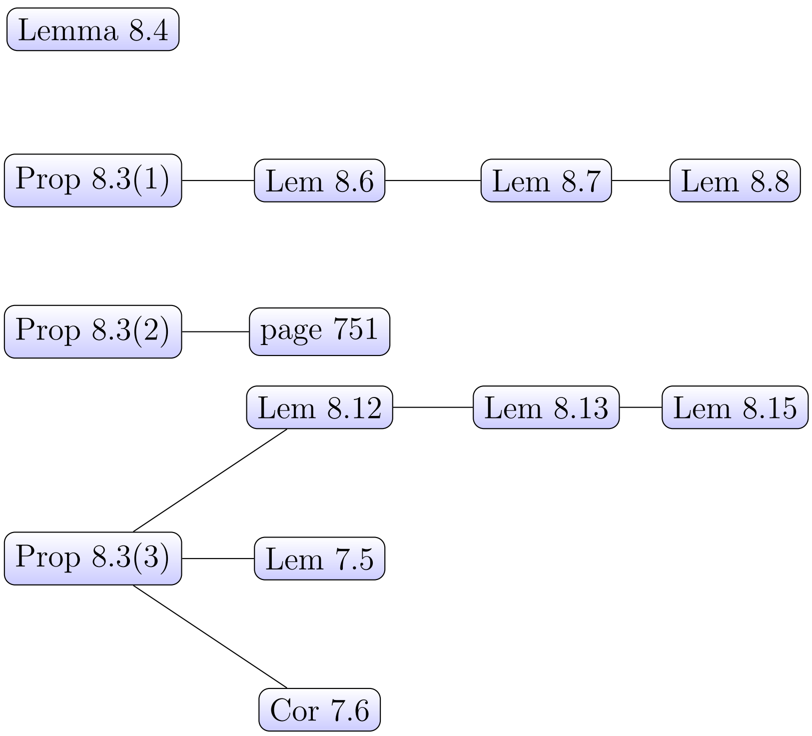


FIGURE 8.1. Structure of Chapter 8.

EXAMPLE 8.1. (i) Let $(Y, y) = (\mathbb{H}, i)$, $A = \mathbb{D}$ and

$$\phi_\lambda(z) = z + \frac{\lambda - i}{\lambda + i}\bar{z}.$$

Then $(\lambda, z) \mapsto \phi$ is a holomorphic motion of $\mathbb{D}$ over $(\mathbb{H}, i)$. We check the conditions:

(1) For a fixed $z \in \mathbb{D}$, $\phi_\lambda(z)$ is analytic in λ.

(2) For a fixed $\lambda \in \mathbb{H}$, $\phi_\lambda(z)$ is injective in z. Namely, if $z, w \in \mathbb{D}$ and $z \neq w$, then $\phi_\lambda(z) = \phi_\lambda(w)$ implies

$$1 = \frac{z - w}{\bar{z} - \bar{w}} = -\frac{\lambda - i}{\lambda + i}.$$

This is impossible since

$$1 = \left|\frac{z - w}{\bar{z} - \bar{w}}\right| > \left|\frac{\lambda - i}{\lambda + i}\right|,$$

since for $\lambda \in \mathbb{H}$ the point i is more near than $-i$.

(3) We have $\phi_i(z) = z$.

A fundamental result is that if ϕ is a holomorphic motion of the whole sphere $\overline{\mathbb{C}}$, then for each fixed $\lambda \in Y$, $\phi_\lambda(z)$ is a quasiconformal map of z.

Given a Riemann sphere with finitely many punctures S with $\#(\overline{\mathbb{C}} \setminus S) \geq 3$, we call a Beltrami coefficient μ on S *harmonic* if

$$\mu(z) = \frac{\psi(\bar{z})}{\varrho_S(z)^2}$$

where $\psi(z)dz^2$ is a holomorphic quadratic differential on S and $\varrho_S(z)|dz|$ is the Poincaré line element[1] in S.

For $\psi(z)dz^2$ to be a holomorphic quadratic differential, ψ just needs to be an analytic function. The Poincaré line element cannot usually be expressed explicitely.

The *modulus* of an annulus

$$A(z_0, r, R) = \{z \in \mathbb{C}\,;\, r < |z - z_0| < R\}$$

is the number $\log R/r$. The modulus is preserved in conformal maps. For example exp maps the rectangle

$$B = \{z = x + iy \in \mathbb{C}\,;\, 0 < x < r \quad 0 < y < 2\pi\}$$

[1]Recall the line elements from the exercises:

$$\mathbb{D}: \frac{|dz|}{1 - |z|^2}, \quad \mathbb{H}: \frac{|dz|}{2\,\mathrm{Im}(z)}, \quad L: \frac{|dz|}{2\cos(\mathrm{Im}(z))}, \quad S: \frac{|dz|\pi/(2\lambda)}{\sin(\pi\,\mathrm{Im}(z)/\lambda)},$$

and

$$A(0,1,R): \frac{|dz|\pi/(2|z|\log R)}{\sin(\pi \log|z|/\log R)}, \quad \mathbb{C}\setminus\mathbb{D}: \frac{|dz|}{2|z|\log|z|}, \quad \mathbb{D}\setminus\{0\}\,;\, \frac{|dz|}{2|z|\log\frac{1}{|z|}}$$

(A) Not ε-thick. (B) Is ε-thick.

FIGURE 8.2. (A) The domain contains an essential annulus. The separated components contain at least two points. (B) The annulus is too thin.

to the annulus $A = A(0, 1, e^r)$. We have $\mathrm{Mod}(B) = \mathrm{Mod}(A) = \log(e^r/1) = r$.

DEFINITION 8.2 (ε-thick). Let $0 < \varepsilon < 1$. A q-pointed sphere $(\overline{\mathbb{C}}, b_1, \ldots, b - q)$ is called ε-thick if there is no annulus $A \subset \overline{\mathbb{C}} \setminus \{b_1, \ldots, b_q\}$ with $\mathrm{Mod}(A) \geq -(1/2\pi)\log\varepsilon$ such that each connected component of $\overline{\mathbb{C}} \setminus A$ contains at least two elements of the set $\{b_1, \ldots, b_q\}$.

The annulus mentioned in 8.2 is called an *essential annulus.*

Let $a_1, \ldots, a_q \in \mathcal{R}_d$, where $d \geq 1$ and $q \geq 3$, be distinct with $a_q \equiv \infty$ and let $x \in X(a_1, \ldots, a_q)$. Over $X(a_1, \ldots, a_q)$, we consider $\{a_1(\lambda), \ldots, a_q(\lambda)\}$ as a holomorphic motion ϕ of q-points $\{a_1(x), \ldots, a_q(x)\} \subset \overline{\mathbb{C}}$. Namely the map

$$\phi : X(a_1, \ldots, a_q) \times \{a_1(x), \ldots, a_q(x)\} \to \overline{\mathbb{C}}$$

is defined by

$$\varphi(\lambda, a_i(x)) = a_i(\lambda).$$

If g is quasimeromorphic, then its *Beltrami* coefficient is

$$\mu_g(z) = \frac{\frac{\partial g}{\partial \bar{z}}}{\frac{\partial g}{\partial z}} = \frac{g_{\bar{z}}}{g_z}$$

PROPOSITION 8.3. *Let f and $x \in \Omega \subset X(a_1, \ldots, a_q)$ be the same as in Proposition 6.3. Assume that $(\overline{\mathbb{C}}, a_1(x), \ldots, a_q(x))$ is $1/2^{20}$-thick.*

(1) There exists a holomorphic motion $\hat{\phi} : \Omega \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which agrees with ϕ on their common domain of definition, such that for each $\lambda \in \Omega$ the Beltrami coefficient $\mu(\hat{\phi}_\lambda)$ is harmonic on $\overline{\mathbb{C}} \setminus \{a_1(x), \ldots, a_q(x)\}$ and satisfies $\|\mu(\hat{\phi}_\lambda)\|_\infty < 1/50$.

(2) We define a map $g : \Omega \to \overline{\mathbb{C}}$ by

$$\hat{\phi}(\lambda, g(\lambda)) = f(\lambda). \tag{8.1}$$

Then g is quasimeromorphic with

$$|\,|g_{\bar{z}}|\,| \leq \frac{1}{50}|g_z|$$

and real-analytic on the inverse image of $\overline{\mathbb{C}} \setminus \{a_1(x), \ldots, a_q(x)\}$.

(3) Let $\Omega^* \Subset \Omega$ be the same as in Proposition 6.3. We recall the notation $\Omega(t)$ from (7.2)*. We have*

$$\int_0^m \left| T\left(r, \frac{f - a_i}{a_j - a_i}, \Omega(t)\right) - T\left(r, \frac{g - a_i(x)}{a_j(x) - a_i(x)}\right) \right| dt$$
$$\leq 2^{29} d q^2 T \left(1 + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4},$$

for $r > \gamma_d$, where i and j are distinct elements in $\{1, \ldots, q-1\}$.

The role of the motion $\hat{\phi}$ in the proof of Proposition 6.3 is to convert the rational targets $a_1, \ldots, a_q$ into constants $a_1(x), \ldots, a_q(x)$, at the price of replacing f by a quasimeromorphic function g. Indeed the two equations $f(z) = a_i(z)$ and $g(z) = a_i(x)$ are equivalent over Ω as the definition (8.1) shows. Thus

$$\bar{n}(g, a_i(x), \Omega(r,t)) = \bar{n}(f, a_i, \Omega(r,t)), \tag{8.2}$$

where we recall $\Omega(r,t) = \Omega(t) \cap \mathbb{C}(r)$. Proposition 8.3(3) claims that the order functions of f and g are close. In the next section, we apply Ahlfors' theory to the quasimeromorphic function g with the constant targets $a_1(x), \ldots, a_q(x)$. The conclusion is Proposition [7.2], which is a main result of Sections 8 and 9.

We remark that Proposition 8.3 is trivial if $q = 3$. Indeed the desired motion is given by a holomorphic map $\hat{\phi} : \Omega \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$\mathrm{cr}(\hat{\phi}(\lambda, z), a_1(\lambda), a_2(\lambda), a_3(\lambda)) = \mathrm{cr}(g(z), a_1(x), a_2(x), a_3(x)).$$

Thus the left-hand side of the estimate of Proposition 8.3(3) is equal to 0. Hence to prove Proposition 8.3, it is enough to consider the case $q \geq 4$.

Teichmüller space. We review some facts from Teichmüller theory which is needed in the proof of Proposition 8.3. For details of the theory, we refer the reader to [**1, 7, 8, 11**]. Let S be a q-punctured sphere, where $q \geq 4$. The Teichmüller space $T(S)$ of S is the set of Teichmüller classes $[\varphi]$ of quasiconformal mappings

$$\varphi : S \to \varphi(S) \subset \overline{\mathbb{C}},$$

where, by definition, two such quasiconformal maps φ and φ' belong to the same Teichmüller class if and only if there exists a conformal map $h : \varphi(S) \to \varphi'(S)$ such that the self mapping $(\varphi')^{-1} \circ h \circ \varphi$ of S is isotopic to the identity modulo the punctures $\overline{\mathbb{C}} \setminus S$. Let S^* be the complex conjugate of S. Let $Q(S^*)$ be the space of holomorphic quadratic differentials on S^* with at worst simple poles at the punctures of S^*. We have the Bers embedding $\beta : T(S) \to Q(S^*)$, which preserves the base points, that is, $\beta([id_S]) = 0$. For each $\psi \in Q(^*)$, we define a harmonic Beltrami differential $\mu[\psi]$ on S by

$$\mu[\psi](z) = -\frac{1}{2}\frac{\psi(\bar{z})}{\varrho_S(z)^2},$$

where $\varrho_S(z)|dz|$ is the Poincaré line element in S. We consider $Q(S^*)$ as a Banach space with a Nehari norm

$$\|\psi\|_\infty \sup_{z\in S} |\mu[\psi](z)|.$$

For $\delta > 0$, we set

$$\mathcal{B}(\delta) = \{\psi \in Q(S^*)\,;\, \|\psi\|_\infty < \delta\}\,.$$

A fundamental result about the Bers embedding [2] is

$$\mathcal{B}(1) \subset \beta(T(S)) \subset \mathcal{B}(3). \tag{8.3}$$

For the Carathéodory distance $c_{\mathcal{B}(3)}$ on $\mathcal{B}(3)$, we have (cf. [10])

$$c_{\mathcal{B}(3)}(0, y) = d_\Delta(0, \|y\|_\infty/3). \tag{8.4}$$

To see this, we remark that for each $z \in S$, the map $y \mapsto \mu[y](z)$ is holomorphic. This gives a holomorphic map $\mu[\cdot](z) : \mathcal{B}(3) \to \Delta(3)$. Hence by the definition of the Carathéodory distance, we have

$$c_{\mathcal{B}(3)}(0, y) \geq \sup_{z\in S} d_\Delta(0, \mu[y](z)/3) = d_\Delta(0, \|y\|\infty/3).$$

On the other hand, there is a holomorphic map $\Delta \to \mathcal{B}(3)$ defined by $t \mapsto (3/\|y\|_\infty)ty$. Thus, by the distance decreasing property, we have

$$d_\Delta(0, \|y\|_\infty/3) \geq c_{\mathcal{B}(3)}(0, y).$$

Thus, we obtain (8.4).

Universal holomorphic motion. Let $E = \{b_1, \ldots, b_3, 0, 1, \infty\} \subset \overline{\mathbb{C}}$ be a set of distinct q-points in the Riemann sphere. Every $t \in T(\overline{\mathbb{C}} \setminus E)$ is a Teichmüller class $[\varphi]$ of a quasiconformal mapping φ of $\overline{\mathbb{C}} \setminus E$ into $\overline{\mathbb{C}}$. Replacing φ by a composition $h \circ \varphi$ with a suitable Möbius transformation h, we may assume without loss of generality that φ is normalized in the sense that φ fixes $0, 1$ and ∞.

The universal holomorphic motion $\Phi : T(\overline{\mathbb{C}} \setminus E) \times E \to \overline{\mathbb{C}}$ over $\Big(T(\overline{\mathbb{C}} \setminus E), [id_{\overline{\mathbb{C}}\setminus E}]\Big)$ is defined by

$$\Phi([\varphi, e]) = \varphi(e),$$

[2]In the case $S = \mathbb{D}$, we would have

$$2|\mu[\psi](z)| = |\psi(\bar{z})|(1 - |z|^2)^2.$$

Recall that $f : \mathbb{D} \to \mathbb{D}$ such that $f'(z) \neq 0$ is *univalent* if $f(z) = f(w)$ implies $z = w$ for $z, w \in \mathbb{D}$. Kraus-Nehari theorem says that if

$$\sup_{z\in\mathbb{D}} |S_f(z)|(1 - |z|^2)^2 \leq 2N$$

holds for $N = 1$, then f is univalent, and if f is univalent, then the condition holds for $N = 3$. These constants 1 and 3 are the same in this context.

where φ is a normalized quasiconformal map. The universal holomorphic motion is well defined. Indeed if φ and φ' are two normalized quasiconformal maps which belong to the same Teichmüller class, then there exists a conformal map h of $\varphi(\overline{\mathbb{C}} \setminus E)$ into $\varphi'(\overline{\mathbb{C}} \setminus E)$ such that $(\varphi')^{-1} \circ h \circ \varphi$ is isotopic to the identity modulo E. The map h is the identity, for h must be a Möbius transform which fixes $0, 1$ and ∞. Thus, we conclude $(\varphi')|_E = \varphi|_E$, which means that the map Φ is well-defined. Note that the universal holomorphic motion is normalized in the sense that $0, 1, \infty \in E$ are the fixed points of the map $\Phi(t, \cdot)$ for every $t \in T(\overline{\mathbb{C}} \setminus E)$.

We extend the universal holomorphic motion Φ to a holomorphic motion $\hat{\Phi} : \mathcal{B}(1) \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of whole sphere over $(\mathcal{B}(1), 0)$. Here, we identify $T(\overline{\mathbb{C}} \setminus E)$ with its image $\beta(T(\overline{\mathbb{C}} \setminus E))$ under the Bers embedding and consider $\mathcal{B}(1) \subset T(\overline{\mathbb{C}} \setminus E)$. The motion is defined by

$$\hat{\Phi}(t, z) = w^{\mu[t]}(z)$$

for $t \in \mathcal{B}(1)$, where $w^{\mu[t]}$ is the normalized quasiconformal mapping whose Beltrami coefficient is $\mu[t]$. By the Ahlfors-Weill theorem, we have $\left[w^{\mu[t]}\right] = t$, hence $\hat{\Phi}(t, e) = \Phi(t, e)$ for all $(t, e) \in \mathcal{B}(1) \times E$. Since $\mu[t]$ depends holomorphically on t, the map $t \mapsto w^{\mu[t]}(z)$ is holomorphic for each fixed $z \in \mathbb{C}$. Thus, $\hat{\Phi}$ is a holomorphic motion of the whole sphere.

We remark that the map $t \mapsto \mu[t](z)$ is holomorphic for each z and the map $(t, z) \mapsto \mu[t](z)$ is real analytic on $\mathcal{B}(1) \times (\overline{\mathbb{C}} \setminus E)$. Hence, by the following lemma, the map $\hat{\Phi} : \mathcal{B}(1) \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is real analytic on $\mathcal{B}(1) \times (\overline{\mathbb{C}} \setminus E)$.

LEMMA 8.4. *Let M be a complex manifold. Let $\nu_t(z) = \nu(t, z)$ be a complex valued function on $M \times \mathbb{C}$ with $|\nu(t, z)| < 1$ such that*

(1) *for each $t \in M$, ν_t is a measurable function on $\mathbb{C}$ and* $\operatorname{ess\,sup}_{z \in \mathbb{C}} |\nu_t(z)| < 1$,
(2) *for each z, the mapping $M \to \Delta$ given by $t \mapsto \nu(t, z)$ is holomorphic,*
(3) *there exists a domain $D \subset \mathbb{C}$ such that ν is real analytic on $M \times D$.*

Then the normalized quasiconformal map $w^{\nu_t}(z)$ is real analytic on $M \times D$.

PROOF. We first construct a local solution $W(t, z) = W_t(z)$ of

$$\frac{\partial}{\partial \bar{z}} W(t, z) = \nu(t, z) \frac{\partial}{\partial z} W(t, z) \tag{8.5}$$

on a neighbourhood of $(t_0, z_0) \in M \times D$ which is injective in z, holomorphic in t and real analytic in (t, z). This is achieved by the Cauchy-Kowalevski theorem. We write as

$$\nu(tz) = \sum_{\alpha,i,j} c_{\alpha,i,j}(t - t_0)^{\alpha}(z - z_0)^i(\bar{z} - \bar{z}_0)^j,$$

where α is multi-index. We set

$$\nu(t, \zeta, \xi) = \sum_{\alpha,i,j} c_{\alpha,i,j}(t - t_0)^{\alpha}(\zeta - z_0)^i(\xi - \bar{z}_0)^j.$$

Then $\eta(t, \zeta, \xi)$ is analytic on a neighbourhood of $(t_0, z_0, \bar{z}_0 \in M \times \mathbb{C} \times \mathbb{C}$. We consider the following differential equation with initial data:

$$\frac{\partial}{\partial \xi}U(t, \zeta, \xi) = \eta(t, \zeta, \xi)\frac{\partial}{\partial \zeta}U(t, \zeta, \xi), \quad U(t, \zeta, \bar{z}_0 = \zeta. \tag{8.6}$$

By the Cauchy-Kowalevski theorem, this equation has unique analytic solution $U(t, \zeta, \xi)$ on a neighbourhood of $U(t_0, z_0, \bar{z}_0)$. The initial data give

$$\frac{\partial}{\partial \zeta}U(t, \zeta, \bar{z}_0) = 1. \tag{8.7}$$

We set $W(t, z) = U(t, z, \bar{z})$. Then $W(t, z)$ is real analytic on a neighbourhood of (t_0, z_0), and holomorphic in t. We note that

$$\frac{\partial}{\partial z}W(t, z) = \frac{\partial}{\partial \zeta}U(t, z, \bar{z}) \quad \frac{\partial}{\partial \bar{z}}W(t, z) = \frac{\partial}{\partial \xi}U(t, z, \bar{z}).$$

Hence, by (8.6), we conclude that $W(t, z)$ is a local solution of (8.5). Also, by (8.7), we obtain $\frac{\partial}{\partial z}W(t, z_0) = 1$. This shows that $W(t, z)$ is injective in z on a possibly smaller neighbourhood of (t_0, z_0). Thus, we have constructed the desired local solution $W(t, z)$ of (8.5).

Next, we set $h(t, z) = h_t(z) = w^{\nu_t} \circ W_t^{-1}(z)$. Then h is holomorphic in z, since w^{ν_t} and W_t have the same complex dilatation. We claim that h is holomorphic in t (cf. [18,p. 242]). To show this, we take a small constant $\varrho > 0$ such that $W_t(z)$ is defined on the closed disc $\{z\,; |z - z_0| \leq \varrho\}$. Let Γ_t be the image under W_t of the circle $z = z_0 + \varrho e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then Γ_t is a smooth Jordan curve, for W_t is real analytic. We note that, by the initial condition in (8.6), $W(t, z_0) = z_0$. Hence z_0 lies in the domain interior to Γ_t. We apply Cauchy's formula to the holomorphic function $h_t(z)$. Then if z is close enough to z_0, we have

$$\begin{aligned}
h_t(z) &= \frac{1}{2\pi i}\int_{\Gamma_t} \frac{h_t(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{h_t \circ W_t(z_0 + \varrho e^{i\theta})}{W_t(z_0 + \varrho(e^{i\theta})) - z}\frac{\partial W_t(z_0 + \varrho(e^{i\theta}))}{\partial \theta}\, d\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{w^{\nu_t}(z_0 + \varrho e^{i\theta})}{W_t(z_0 + \varrho(e^{i\theta})) - z}\frac{\partial W_t(z_0 + \varrho(e^{i\theta}))}{\partial \theta}\, d\theta.
\end{aligned}$$

Since the functions $w^{\nu_t}(z_0+\varrho(e^{i\theta}))$, $W_t(z_0+\varrho(e^{i\theta}))$, $\partial W_t(z_0+\varrho(e^{i\theta}))/d\theta$ are holomorphic in t and continuous in (t,θ), we conclude that the map $t\mapsto h_t(z)$ is holomorphic. Hence by Hartogs' theorem, h is holomorphic in (t,z). Hence $w^{\nu_t}(z)$ is real analytic on $M\times D$. □

THEOREM 8.5 (Hartogs). *Let $F:\mathbb{C}^n\to\mathbb{C}$ be analytic in each variable z_j, for $j=1,\dots,n$, when other variables are constant. Then:*
(i) F is continuous;
(ii) F is analytic in n-variable sense: each $x\in C^n$ has an open neighbourhood U such that

$$F(z)=\sum_{m=1}^{\infty}\sum_{k_1+\cdots+k_n\le m}a(k_1,\dots,k_n)(z_1-x)^{k_1}\cdot(z_n-x)^{k_n},$$

for $z\in U$.

8.0.1. Proof of Proposition 8.3(1). We prove a more general statement.

LEMMA 8.6. *Let Ω be a neighbourhood of $x\in X(a_1,\dots,a_q)$. Assume that one of the following conditions is true:*

1. *Ω is a topological disk with $\ell_X(a_1,\dots,a_q)(\partial\Omega)<1/75$, or*
2. *Ω is an annulus with $\ell_{X(a_1,\dots,a_q)}<\varepsilon/(25q)$ and $(\overline{\mathbb{C}},a_1(x),\dots,a_q(x))$ is ε-thick, where $0<\varepsilon<1$.*

Then there exists a holomorphic motion $\hat\phi:\Omega\times\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ which agrees with ϕ on their common domain of definition, such that for each $\lambda\in\Omega$ the Beltrami coefficient $\mu(\hat\phi_\lambda)$ is harmonic on $\overline{\mathbb{C}}\setminus\{a_1(x),\dots,a_q(x)\}$ and satisfies $\|\mu(\hat\phi_\lambda)\|_\infty<1/50$.

We follow the proof of Bers-Royden's $\frac{1}{3}$-extension theorem cf. [4]. To normalize the motion ϕ, we set $\alpha_i(z)=\mathrm{cr}(a_i(z),a_{q-2}(z),a_{q-1}(z),a_q(z))$ for $i=1,\dots,q$. Then $\alpha_{q-2}=0$, $\alpha_{q-1}=1$ and $\alpha_q=\infty$. Set $E=\{\alpha_1(x),\dots,\alpha_{q-3},0,1,\infty\}$. We denote by φ the holomorphic motion

$$\{\alpha_1(z),\dots,\alpha_{q-3},0,1,\infty\}$$

of E over (X,x), where we write $X=X(a_1,\dots,a_q)$ to simplify the notation.

We denote by $\mathcal{M}_{0,q}$ the complex manifold of ordered $(q-3)$-tuples of distinct complex numbers $(c_1,\dots,c_{q-3})$ none of which equals 0 or 1. Using the universal holomorphic motion Φ, we may define a holomorphic map $p:T(\overline{\mathbb{C}}\setminus E)\to\mathcal{M}_{0,q}$ by

$$t\mapsto(\Phi(t,\alpha_1(x)),\dots,\Phi(t,\alpha_{q-3}(x))).$$

The map p is a universal covering map [4,p. 268]. We consider the motion φ as a holomorphic map $\boldsymbol{\varphi}:X\to\mathcal{M}_{0,q}$ defined by

$$\boldsymbol{\varphi}(z)=(\alpha_1(z),\dots,\alpha_{q-3}(z)).\tag{8.8}$$

The key lemma to prove Lemma 8.6 is as follows:

LEMMA 8.7. *Assume that Ω satisfies the assumption of Lemma 8.6. Then there exists a lifting $\tilde{\boldsymbol{\varphi}} : \Omega \to T(\overline{\mathbb{C}} \backslash E)$ of $\boldsymbol{\varphi}$ over Ω which preserves the base points $\tilde{\boldsymbol{\varphi}}(x) = [\mathrm{id}]$. Moreover, we have $\tilde{\boldsymbol{\varphi}}(\Omega) \subset \mathcal{B}(1/50)$.*

If we assume this lemma, we may prove Lemma 8.6 as follows. We first remark that over Ω, the motion φ is the pull-back of the universal motion Φ by $\tilde{\boldsymbol{\varphi}}$. Namely for $(\lambda, e) \in \Omega \times E$, we have

$$\varphi(\lambda, e) = \Phi(\tilde{\boldsymbol{\varphi}}(\lambda), e).$$

We define a holomorphic motion $\hat{\varphi} : \Omega \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the whole sphere by

$$\hat{\varphi}(\lambda, z) = \hat{\Phi}(\tilde{\boldsymbol{\varphi}}(\lambda), z) \tag{8.9}$$

for $(\lambda, z) \in \Omega \times \overline{\mathbb{C}}$. Then $\hat{\varphi}$ is an extension of φ. The Beltrami coefficient satisfies $\mu(\hat{\phi}_\lambda) = \mu[\tilde{\boldsymbol{\varphi}}(\lambda)]$ for each $\lambda \in \Omega$. Hence, by $\tilde{\boldsymbol{\varphi}}(\Omega) \subset \mathcal{B}(1/50)$, we obtain $\|\mu(\hat{\varphi}_\lambda)\|_\infty < 1/50$ for each $\lambda \in \Omega$.

Now the holomorphic motion $\hat{\phi} : \Omega \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$\mathrm{cr}(\hat{\phi}(\lambda, z), a_{q-2}(\lambda), a_{q-1}(\lambda), a_q(\lambda)) = \hat{\varphi}(\lambda, \mathrm{cr}(z, a_{q-2}(x), a_{q-1}(x), a_q(x))) \tag{8.10}$$

has the desired properties. Thus we have derived Lemma 8.6 from Lemma 8.7.

It remains to prove Lemma 8.7. For $y \in \mathcal{M}_{0,q}$, we set

$$B_\delta(y) = B_{\mathcal{M}_{0,q}}(y, \delta) = \left\{ w \in \mathcal{M}_{0,q} \, ; \, d_{\mathcal{M}_{0,q}}(y, w) < \delta \right\},$$

where $d_{\mathcal{M}_{0,q}}$ is the Kobayashi-Teichmüller distance on $\mathcal{M}_{0,q}$.

LEMMA 8.8. *Let $y = (y_1, \ldots, y_{q-3}) \in \mathcal{M}_{0,q}$ be a point such that $(\overline{\mathbb{C}}, y_1, \ldots, y_{q-3}, 0, 1, \infty)$ is ε-thick, where $0 < \varepsilon < 1$. Then $B_{\mathcal{M}_{0,q}}(y, e/(50q))$ has an injective lift to the universal covering $p : T(\overline{\mathbb{C}} \setminus E) \to \mathcal{M}_{0,q}$.*

Lemma 8.8 has been improved by Hiroshige Shiga[3]
Yamanoi's result reads

THEOREM 8.9. *For any $p \in \mathcal{M}_n$,*

$$\frac{\mathcal{E}_p}{50n} \leq r_q(\mathcal{M}_n)$$

holds, if there is no essential annulus A in $X(p)$ with $Mod(A) \geq -\frac{1}{2\pi} \log \mathcal{E}_p$.

Let $p = \{p_1, \ldots, p_n\}$ and $X(p) = \overline{\mathbb{C}} \setminus \{p_1, \ldots, p_n\}$.
An annulus $A \subset X(p)$ is an essential annulus if the components $B \cup C$ of

$$X(p) \setminus A = B \cup C$$

satisfy $\# p \cap B \geq 2$ and $\# p \cap C \geq 2$.

[3]Shiga, On injectivity radius in configuration space and in moduli space, Contemporary Mathematics Volume 590, 2013.

If $X(p)$ contains no essential annulus A such that $Mod(A) \geq -\frac{1}{2\pi}\log\varepsilon$, then $X(p)$ is ε-thick.

Shiga has shown

THEOREM 8.10. *For any $p \in \mathcal{M}_n$,*

$$\min\left\{\log(2+\sqrt{5}), \log\sqrt{\left(\frac{\ell_p}{\pi}\right)^2+1}\right\} \leq r_p(\mathcal{M}_n)$$

holds, where ℓ_p is the length of the shortest closed geodesic in $X(p)$.

For the moduli space, Shiga has shown

THEOREM 8.11. *Let X_0 be a Riemann surface of type (g,n) with $2g-2+n>0$. Then, for any $p=[X_p,f_p]\in M(X_0)$, we have*

$$M(g,n)^{-1}\min\left\{\log 2, \log\left\{\frac{\ell_p^2}{\pi^2}+1\right\}\right\} \leq r_P(M(X_0)),$$

where $M(g,n)=\{2\cdot 84(g-1)+4n\}\,(2g-2+n)!$ and ℓ_p is the length of the shortest closed geodesic in X_p.

Here the constant $84(g-1)$ is the same as in Hurwitz' automorphism theorem.

PROOF. Note that $\mathcal{M}_{0,q}$ is a domain of $\mathbb{C}^{q-3}$. Using the point $y\in\mathcal{M}_{0,q}$, we define a domain $P(y)\subset\mathbb{C}^{q-3}$ by the following rule: $b=(b_1,\dots,b_{q-3})\in P(y)$ if and only if

$$\operatorname{Re}\left(\frac{b_i}{y_i}\right)>0,\quad \operatorname{Re}\left(\frac{b_i-1}{y_i-1}\right)>0,\quad \operatorname{Re}\left(\frac{b_i-b_j}{y_i-y_j}\right)>0$$

for $1\leq i\leq q-3$ and $1\leq j\leq q-3$ with $j\neq i$. Then by the definition, we immediately conclude that

$$P(y)\subset\mathcal{M}_{0,q}.$$

Next, we remark that $P(y)$ is convex. Indeed if $b=(b_1,\dots,b_{q-3})$ and $c=(c_1,\dots,c_{q-3})\in P(y)$, we have

$$(tb_1+(1-t)c_1,\dots,tb_{q-3}+(1-t)c_{q-3})\in P(y)$$

for $0\leq t\leq 1$. This follows from

$$\operatorname{Re}((a+b)c)=\operatorname{Re}(ac)+\operatorname{Re}(bc),\quad \operatorname{Re}(\alpha a)=\alpha\operatorname{Re}(a),$$

for $a,b,c\in\mathbb{C}$ and $\alpha\in\mathbb{R}$, which implies that

$$\operatorname{Re}\left(\frac{tb_i+(1-t)c_i}{y_i}\right)=t\operatorname{Re}\left(\frac{b_i}{y_i}\right)+(1-t)\operatorname{Re}\left(\frac{c_i}{y_i}\right)>0,$$

$$\operatorname{Re}\left(\frac{tb_i+(1-t)c_i-1}{y_i-1}\right)=t\operatorname{Re}\left(\frac{b_i-1}{y_i}-1\right)+(1-t)\operatorname{Re}\left(\frac{c_i-1}{y_i-1}\right)>0,$$

$$\operatorname{Re}\left(\frac{tb_i+(1-t)c_i-tb_j-(1-t)c_j}{y_i-y_j}\right)=t\operatorname{Re}\left(\frac{b_i-b_j}{y_i-y_j}\right)+(1-t)\operatorname{Re}\left(\frac{c_i-c_j}{y_i-y_j}\right)>0.$$

Now $P(y)$ is convex, hence simply connected. Thus there exists an injective lift $P(y) \subset T(\overline{\mathbb{C}} \setminus E)$ to the universal covering $p : T(\overline{\mathbb{C}} \setminus E) \to \mathcal{M}_{0,q}$.

We finish the proof by showing $B_{\mathcal{M}_{0,q}}(y, \varepsilon/50q) \subset P(y)$. For distinct i, j, k and l in $\{1, \dots, q\}$, we define a holomorphic map $\eta[i, j, k, l] : \mathcal{M}_{0,q} \to \mathbb{C} \setminus \{0, 1\}$ by

$$\eta[i, j, k, l]((b_1, \dots, b_{q-3})) = \mathrm{cr}(b_i, b_j, b_k, b_l),$$

where $(b_1, \dots, b_{q-3}) \in \mathcal{M}_{0,q}$ and we set $b_{q-2} = 0$, $b_{q-1} = 1$, $b_q = \infty$.

Claim. Let $b = (b_1, \dots, b_{q-3} \in B(y, \varepsilon/50q)$. Then we have

$$\left| \arg \frac{\eta[i, j, k, l](b)}{\eta[i, j, k, l](y)} \right| < \frac{\pi}{4}$$

for all distinct i, j, k and l in $\{1, \dots, q\}$.

Proof. We assume without loss of generality that $|\eta[i, j, k, l](y)| \leq 1$, for otherwise we replace $\eta[i, j, k, l]$ by $\eta[i, j, k, l]^{-1} = \eta[i, l, k, j]$. We first prove that if $\varepsilon \leq |\eta[i, j, k, l](y)| \leq 1$, then

$$\left| \arg \frac{\eta[i, j, k, l](b)}{\eta[i, j, k, l](y)} \right| < \frac{\pi}{4q}. \tag{8.11}$$

To show this, we take a hyperbolic geodesic γ connecting $\eta[i, j, k, l](y)$ and $\eta[i, j, k, l](b)$ in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then by the distance decreasing property, we have $\ell_{\overline{\mathbb{C}} \setminus \{0,1,\infty\}}(\gamma) < \varepsilon/(50q)$. We apply (7.5) to obtain

$$\ell\overline{\mathbb{C}}(\gamma) < \frac{\varepsilon}{10q}.$$

This shows that $\gamma \subset \{|z| \leq 2\}$. Hence, we obtain

$$|\eta[i, j, k, l] - \eta[i, j, k, l](b)| \leq \ell_{\mathrm{Euclid}}(\gamma) \leq 5\ell_{\overline{\mathbb{C}}}(\gamma) < \frac{\varepsilon}{2q},$$

where

$$\ell_{\mathrm{Euclid}}(\gamma) = \int_\gamma |dz|.$$

Thus,

$$\left| \arg \frac{\eta[i, j, k, l](b)}{\eta[i, j, k, l](y)} \right| < \frac{\pi}{2}$$

and

$$\sin \left| \arg \frac{\eta[i, j, k, l](b)}{\eta[i, j, k, l](y)} \right| < \frac{1}{2q}.$$

Hence, we obtain (8.11).

Next, we consider the general case. We order all numbers x in

$$\{|\,\mathrm{cr}(y_1, y_j, y_k, y_l)|, \dots, |\,\mathrm{cr}(y_q, y_j, y_k, y_l)|\}$$

that satisfy

$$|\,\mathrm{cr}(y_1, y_j, y_k, y_l)| \leq x \leq 1$$

in the form

$$|\operatorname{cr}(y_{i_1}, y_j, y_k, y_l)| \le |\operatorname{cr}(y_{i_2}, y_j, y_k, y_l)| \le \ldots \le |\operatorname{cr}(y_{i_s}, y_j, y_k, y_l)| = 1,$$

where $i_1 = i, i_s = k$ and $s \le q$. Since $(\overline{\mathbb{C}}, y_1, \ldots, y_q)$ is ε-thick, we conclude that

$$\varepsilon \le \frac{|\operatorname{cr}(y_{i(t)}, y_j, y_k, y_l)|}{|\operatorname{cr}(y_{i(t+1)}, y_j, y_k, y_l)|} \le 1 \tag{8.12}$$

for $t = 1, \ldots, s-1$; otherwise, the annulus

$$\{z\,;\, |\operatorname{cr}(y_{i(t+1)}, y_j, y_k, y_l)|\varepsilon < |z| < |\operatorname{cr}(y_{i(t+1)}, y_j, y_k, y_l)|\}$$

separates the q-points

$$\operatorname{cr}(y_1, y_j, y_k, y_l), \operatorname{cr}(y_2, y_j, y_k, y_l), \ldots, \operatorname{cr}(y_q, y_j, y_k, y_l),$$

which is a contradiction. Since

$$\operatorname{cr}(y_{i(t)}, y_j, y_{i(t+1)}, y_l) = \frac{\operatorname{cr}(y(i(t), y_j, y_k, y_l)}{\operatorname{cr}(y(i(t+1), y_j, y_k, y_l)},$$

we have

$$\eta[i_t, j, i_{t+1}, l] = \frac{\eta[i_t, j, k, l]}{\eta[i_{t+1}, j, k, l]}. \tag{8.13}$$

Thus by (8.12) and (8.13), we conclude for $t = 1, \ldots, s-1$,

$$\varepsilon \le |\eta[i_t, j, i_{t+1}, l](y)| \le 1,$$

hence by (8.11)

$$\left|\arg \frac{\eta[i_t, j, i_{t+1}, l](b)}{\eta[i_t, j, i_{t+1}, l](y)}\right| < \frac{\pi}{4q}. \tag{8.14}$$

Using (8.13) again, we obtain

$$\left|\arg \frac{\eta[i_t, j, k, l](b)}{\eta[i_t, j, i_k, l](y)} - \arg \frac{\eta[i_{t+1}, j, i_k, l](b)}{\eta[i_{t+1}, j, k, l](y)}\right| < \frac{\pi}{4q}.$$

Summing both sides of this estimate for $t = 1, \ldots, s-2$ and (8.14) for $t = s-1$, we establish our claim. $\square$

Now we go back to the proof of Lemma 8.8. Let $b = (b_1, \ldots, b_{q-3}) \in B_{\mathcal{M}_{0,q}}(y, \varepsilon/50q)$. We have

$$\eta[i, q-2, q-1, q](b) = b_i,$$
$$\eta[i, q-1, q-2, q](b) = 1 - b_i,$$
$$\eta[j, i, q-2, q](b) = \frac{b_i - b_j}{b_i}.$$

Hence, by the claim above, we have[4]

$$\left|\arg\left(\frac{b_i}{y_i}\right)\right| < \frac{\pi}{4},$$

$$\left|\arg\left(\frac{1-b_i}{1-y_i}\right)\right| < \frac{\pi}{4},$$

$$\left|\arg\left(\frac{y_i}{y_i-y_j}\frac{b_i-b_j}{b_i}\right)\right| < \frac{\pi}{4},$$

where $1 \le i \le q-3$, $1 \le j \le q-3$ and $i \ne j$. Thus, we conclude $(b_1, \dots, b_{q-3}) \in P(y)$, hence $B_{\mathcal{M}_{0,q}}(y, \varepsilon/50q) \subset P(y)$. This concludes the proof of Lemma 8.8 $\square$

Hiroshige Shiga has improved Lemma 8.8, see Contemporary Math. 590, p. 183.

Proof of Lemma 8.7. We set $X_\Omega = \widetilde{X}/\operatorname{Im}(\pi_1(\Omega) \to \pi_1(X))$, where $\widetilde{X}$ is the universal covering of X. Then $\Omega \subset X_\Omega$. Note that X_Ω is an annulus when $\Omega \subset X$ is an essential annulus; otherwise, X_Ω is a disk.

We show that there is a lift $\tilde{\boldsymbol{\varphi}} : \Omega \to T(\overline{\mathbb{C}} \setminus E)$ of $\boldsymbol{\varphi}$ over Ω. Let $b : X_\Omega \to \mathcal{M}_{0,q}$ be the composition of the covering map $X_\Omega \to X$ and $\boldsymbol{\varphi}$. Then it is enough to show the existence of a lift $\widetilde{b} : X_\Omega \to T(\overline{\mathbb{C}} \setminus E)$ of b with $\widetilde{b}(x) = [\mathrm{id}]$. If X_Ω is a disc, this is obvious. Assume that X_Ω is an annulus. there exists an essential loop γ in X_Ω passing through x with $\ell_{X_\Omega}(\gamma) < \varepsilon/(25q)$. Then by the distance decreasing property, we have $b(\gamma) \subset B_{\mathcal{M}_{0,q}}(\boldsymbol{\varphi}(x), \varepsilon/50q)$. Hence, by Lemma 8.8, we conclude the existence of the lift $\widetilde{b} : X_\Omega \to T(\overline{\mathbb{C}} \setminus E)$ with $b(x) = [\mathrm{id}]$.

Next we show $\|\beta \circ \widetilde{b}(y)\| < 1/50$ for all $y \in \Omega$, where $\beta : T(\overline{\mathbb{C}} \setminus E) \to Q((\overline{\mathbb{C}} \setminus E)^*)$ is the Bers embedding. Since the hyperbolic length of the boundary of $\Omega \subset X_\Omega$ is less than 1/75, we have

$$c_{X_\Omega}(x, y) < \frac{1}{150} \tag{8.15}$$

for $y \in \Omega$, where c_{X_Ω} is the Carathéodory distance of X_Ω. We note that the image of the map $\beta \circ \widetilde{b} : X_\Omega \to Q((\overline{\mathbb{C}} \setminus E)^*)$ is contained in $\mathcal{B}(3)$ (cf. (8.3)). Thus using (8.4), we have for $y \in \Omega$

$$c_{X_\Omega}(x, y) \ge c_{\mathcal{B}(3)}(0, \beta \circ \widetilde{b}(y)) = d_\Delta(0, \|\beta \circ \widetilde{b}(y)\|_\infty/3) \ge \|\beta \circ \widetilde{b}(y)\|_\infty/3.$$

Thus by (8.15), we have $\|\beta \circ \widetilde{b}(y)\|_\infty < 1/50$. Hence, we conclude $\widetilde{b}(\Omega) \subset \mathcal{B}(1/50)$. $\square$

8.0.2. Proof of Proposition 8.3(2). Set

$$\begin{aligned} \widetilde{f} &= \operatorname{cr}(f, a_{q-2}, a_{q-1}, a_q) \\ \widetilde{g} &= \operatorname{cr}(g, a_{q-2}(x), a_{q-1}(x), a_q(x)). \end{aligned} \tag{8.16}$$

[4] by the properties of the cross-ratio

Then by (8.1) and (8.10), we have

$$\hat{\varphi}(\lambda, \widetilde{g}(\lambda)) = \widetilde{f}(\lambda).$$

Hence, by (8.9), we have

$$\widetilde{f}(\lambda) = \hat{\Phi}(\tilde{\boldsymbol{\varphi}}(\lambda), \widetilde{g}(\lambda)). \tag{8.17}$$

Let $\Psi : \mathcal{B}(1) \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined by

$$\hat{\Phi}(y, \Psi(y, z)) = z$$

for each $(y, z) \in \mathcal{B}(1) \times \overline{\mathbb{C}}$. Then we have

$$\widetilde{g}(\lambda) = \Psi(\tilde{\boldsymbol{\varphi}}(\lambda), \widetilde{f}(\lambda)). \tag{8.18}$$

Since the Jacobian of $\hat{\Phi}$ does not vanish (cf. [24,p. 37]), Ψ is real analytic outside $\Psi^{-1}(E)$. Hence g is real-analytic on the inverse image of $\overline{\mathbb{C}} \setminus \{a_1(x), \ldots, a_q(x)\}$.

By (8.1), we have

$$\frac{\partial f}{\partial \overline{\lambda}}(\lambda) = \hat{\phi}_{\overline{\lambda}}(\lambda, g(\lambda)) + \hat{\phi}_z(\lambda, g(\lambda))\frac{\partial g}{\partial \overline{\lambda}} + \hat{\phi}_{\overline{z}}(\lambda, g(\lambda))\frac{\partial \overline{g}}{\partial \overline{\lambda}}(\lambda).$$

Since $f(\lambda)$ is holomorphic, we have $\frac{\partial f}{\partial \overline{\lambda}} = 0$. Since $\hat{\phi}$ is holomorphic in λ, we have $\hat{\phi}_{\overline{\lambda}}(\lambda, g(\lambda)) = 0$. Hence, we obtain

$$\hat{\phi}_z(\lambda, g(\lambda))\frac{\partial g}{\partial \overline{\lambda}}(\lambda) + \hat{\phi}_{\overline{z}}(\lambda, g(\lambda))\frac{\partial \overline{g}}{\partial \overline{\lambda}}(\lambda) = 0.$$

Hence, we have

$$\left|\frac{g_{\overline{\lambda}}(\lambda)}{g_{\lambda}(\lambda)}\right| = \left|\frac{g_{\overline{\lambda}}(\lambda}{\overline{g}_{\overline{\lambda}}(\lambda)}\right| = \left|\frac{\hat{\phi}_{\overline{z}}(\lambda, g(\lambda))}{\hat{\phi}_z(\lambda, g(\lambda))}\right| < \frac{1}{50}.$$

This shows Proposition 8.3(2). □

Proof of Proposition 8.3(3) We may assume wihtout loss of generality that $i = q-2$ and $j = q-1$. Thus with the previous notation (8.16), we are going to prove

$$\begin{aligned} &\int_0^m |T(r, \widetilde{f}, \Omega(t)) - T(r, \widetilde{g}, \Omega(t))|\, dt \\ &\le 2^{29} d q^2 T\left(1 + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4} \end{aligned} \tag{8.19}$$

for $r > \gamma_d$. Here, $\widetilde{f}$ and $\widetilde{g}$ satisfy (8.17) as well as (8.18).

For $(y, z) \in \mathcal{B}(1/50) \times \overline{\mathbb{C}}$, we set

$$H(y, z) = \int_{\overline{\mathbb{C}}} \eta(z, \hat{\Phi}(y, w)), \omega_{\overline{\mathbb{C}}}[w] - \int_{\overline{\mathbb{C}}} \eta(\infty, \hat{\Phi}(y, w)), \omega_{\overline{\mathbb{C}}}[w],$$

where

$$\eta(z, z') = -\log[z, z']^2$$

for $z, z' \in \overline{\mathbb{C}}$. By the Hölder continuity of the quasiconformal map $\hat{\Phi}(y, \cdot)$, the two integrals in the definition of $H(y, z)$ are bounded. See the remark after Lemma 8.13.

The key lemma in the proof of Proposition 8.3(3) is as follows:

LEMMA 8.12. *(1) On $(\mathcal{B}(1/50) \times \overline{\mathbb{C}}) \setminus \Psi^{-1}(E)$, H is smooth and satisfies*

$$dd^c H = p_2^* \omega_{\overline{\mathbb{C}}} \setminus \Psi^* \omega \overline{\mathbb{C}}, \tag{8.20}$$

where $p_2 : \mathcal{B}(1/50) \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the second projection.

(2) Let $(y, z) \in (\mathcal{B}(1/50) \times \overline{\mathbb{C}}) \setminus \Psi^{-1}(E)$ and let $v = (v_1, v_2) \in T_{(y,z)}(\mathcal{B}(1/50) \times \overline{\mathbb{C}}$ be a tangent vector. Then we have

$$|d^c H(v)| \leq 2^{24}(\|v_1\|_{T(\overline{\mathbb{C}} \setminus E)} + \|v_2\|_{\overline{\mathbb{C}}}),$$

where $\|v_1\|_{T(\overline{\mathbb{C}} \setminus E}$ is the infinitesimal Kobayashi metric on $T(\overline{\mathbb{C}} \setminus E)$ and $\|v_2\|_{\overline{\mathbb{C}}}$ is the spherical line element on $\overline{\mathbb{C}}$.

Derivation of Proposition 8.3 from Lemma 8.12. We consider the holomorphic map $(\tilde{\boldsymbol{\varphi}}, \widetilde{f}) : \Omega \to \mathcal{B}(1/50) \times \overline{\mathbb{C}}$ and the composite function $H(\tilde{\boldsymbol{\varphi}}(\lambda), \widetilde{f}(\lambda))$ defined over $\lambda \in \Omega$. By Lemma 8.12(1), $H(\tilde{\boldsymbol{\varphi}}(\lambda), \widetilde{f}(\lambda))$ is smooth outside $(\tilde{\boldsymbol{\varphi}}, \widetilde{f})^{-1}(\Phi^{-1}(E))$. Here, $(\tilde{\boldsymbol{\varphi}}, \widetilde{f})^{-1}(\Phi^{-1}(E))$ is the set of λ with $\hat{\Phi}(\tilde{\boldsymbol{\varphi}}(\lambda), E) = \widetilde{f}(\lambda)$, which is a discrete set on Ω. We denote by $\Omega_\varepsilon(t)$ a subdomain of $\Omega(t) \subset \mathbb{C}$ obtained by deleting ε-neighbourhood, in the Euclidean distance, of the points where $H(\tilde{\boldsymbol{\varphi}}(\lambda), \widetilde{f}(\lambda))$ is not smooth.

Now by (8.18) and (8.20), we have

$$dd^c H(\tilde{\boldsymbol{\varphi}}, \widetilde{f}) = \widetilde{f}^* \omega_{\overline{\mathbb{C}}} - \widetilde{g}^* \omega_{\overline{\mathbb{C}}},$$

hence

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(r,t)} dd^c H(\tilde{\boldsymbol{\varphi}}, \widetilde{f}) = \int_{\Omega(r,t)} \widetilde{f}^* \omega_{\overline{\mathbb{C}}} - \int_{\Omega(r,t)} \widetilde{g}^* \omega_{\overline{\mathbb{C}}}.$$

Using Stoke's formula and Lemma 8.12(2), we obtain

$$\begin{aligned}
&\left| \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} dd^c H(\tilde{\boldsymbol{\varphi}}, \widetilde{f}) \right| \\
&= \left| \lim_{\varepsilon \to 0} \int_{\partial \Omega_\varepsilon} d^c H(\tilde{\boldsymbol{\varphi}}, \widetilde{f}) \right| \\
&\leq \lim_{\varepsilon \to 0} 2^{24} \left(\ell_{T(\overline{\mathbb{C}} \setminus E}(\tilde{\boldsymbol{\varphi}}(\partial \Omega(r,t)) + \ell_{\overline{\mathbb{C}}}(\widetilde{f}(\partial \Omega(r,t))) \right),
\end{aligned}$$

where $\ell_{T(\overline{\mathbb{C}} \setminus E}$ is the length function with respect to the infinitesimal metric $\|\cdot\|_{T(\overline{\mathbb{C}} \setminus E}$. Since

$$\ell_{T(\overline{\mathbb{C}} \setminus E}(\tilde{\boldsymbol{\varphi}}(\partial \Omega(r,t))) \leq \ell_X(\partial \Omega(r,t)),$$

we obtain

$$\left|\int_{\Omega(r,t)} \widetilde{f}^*\omega_{\overline{\mathbb{C}}} - \int_{\Omega(r,t)} \widetilde{g}^*\omega_{\overline{\mathbb{C}}}\right| \leq 2^{24}(\ell_X(\partial\Omega(r,t)) + \ell_{\overline{\mathbb{C}}}(\widetilde{f}(\partial\Omega)))).$$

Taking the integral on both sides, we have

$$\begin{aligned}&\int_0^m |T(r,\widetilde{g},\Omega(t)) - T(r,\widetilde{f},\Omega(t))|\, dt \\ &\leq 2^{24}\int_0^m \int_1^r (\ell_X(\partial\Omega(r,t)) + \ell_{\overline{\mathbb{C}}}(\widetilde{f}(\partial\Omega(u,t))))\frac{du}{u}dt.\end{aligned}$$

To estimate the right-hand side, we remark that

$$\int_0^m \int_1^r \ell_X(\partial\Omega(r,t))\frac{du}{u}dt \leq 2^4 dq^2 \log r.$$

Indeed, by (6.3), we may apply Lemma 7.5 to the case $\Lambda(r) = 2dq^2\log r$ to obtain this estimate. By Corollary 7.6, we obtain the estimate 8.19. Thus, we have derived Proposition 8.3(3) from Lemma 8.12. □

It remains to prove Lemma 8.12. We begin with the following lemma:

LEMMA 8.13. *Let $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a K-quasiconformal map which fixes $0, 1$ and ∞. Suppose that $K < 2$. Then, for each $z \in \overline{\mathbb{C}}$, we have*[5]

$$\int_{\overline{\mathbb{C}}} \frac{1}{[z,\psi(w)]}\omega_{\overline{\mathbb{C}}}[w] < \pi + 2^{14}\pi\frac{K}{2-K}.$$

REMARK 8.14. Since $K(\hat{\Phi}(y,\cdot) < \frac{51}{49}$ for $y \in \mathcal{B}(1/50)$, we conclude

$$\int_{\overline{\mathbb{C}}} \frac{1}{[z,\hat{\Phi}(y,w)]}\omega_{\overline{\mathbb{C}}}[w] < 2^{18} \tag{8.21}$$

for $(y,z) \in \mathcal{B}(1/50) \times \overline{\mathbb{C}}$. Hence, we have

$$\int_{\overline{\mathbb{C}}} \eta(z,\hat{\Phi}(y,w))\omega_{\overline{\mathbb{C}}}[w] \leq 2\int_{\overline{\mathbb{C}}} \frac{1}{[z,\hat{\Phi}(y,w)]}\omega_{\overline{\mathbb{C}}}[w] < 2^{19}.$$

Thus the integrals in the definition of $H(y,z)$ are bounded.

Proof of Lemma 8.13. We consider the inverse map $\psi^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which is quasi-conformal map fixing $0, 1, \infty$ and satisfying $K(\psi^{-1}) = K(\psi)$. We set

$$\phi(r)\int_{\psi^{-1}(D_x(r)}\omega_{\overline{\mathbb{C}}},$$

where $D_z(r) = \{w \in \overline{\mathbb{C}}\,;\, [w,z] < r\}$. Then we have

$$\int_{\overline{\mathbb{C}}} \frac{1}{[z,\psi(w)]}\omega_{\overline{\mathbb{C}}}[w] = \int_0^1 \frac{1}{r}d\varphi(r).$$

[5]What happens if $K = 1$?

Since

$$[\psi^{-1}(z), \psi^{-1}(w)] \leq 128[z,w]^{1/K}(cf.[5,$$

Lemma 4.1]), we have

$$\psi^{-1}(D_z(r)) \subset D_{\psi^{-1}(z)}(128r^{1/K}).$$

Hence, we have

$$\varphi(r) \leq \int_{D_{\psi^{-1}(z)}(128r^{1/K})} \omega_{\overline{\mathbb{C}}} = 2^{14}\pi r^{2/K}.$$

Since $K < 2$, we obtain

$$\begin{aligned}
\int_{\overline{\mathbb{C}}} \frac{1}{[z,\psi(w)]}\omega_{\overline{\mathbb{C}}}[w] &= \int_0^1 \frac{1}{r} d\varphi(r) \\
&= \lim_{\delta\to 0} \int_\delta^1 \frac{1}{r} d\varphi(r) \\
&= \lim_{\delta\to 0} \left(\left[\frac{1}{r}\varphi(r)\right]_{r=\delta}^1 + \int_\delta^1 \frac{1}{r^2}\varphi(r)dr \right) \\
&\leq \lim_{\delta\to 0} \left(\varphi(1) - \frac{\varphi(\delta)}{\delta} + 2^{14}\pi \int_\delta^1 r^{\frac{2}{K}-2} dr \right) \\
&= \pi + 2^{14}\pi \frac{K}{2-K} \lim_{\delta\to 0} \left[r^{2/K-1}\right]_{r=\delta}^1 \\
&= \pi + 2^{14}\pi \frac{K}{2-K}.
\end{aligned}$$

□

Next we show that H is Lipschitz continuous.

LEMMA 8.15. *For* $(y,z), (y',z') \in \mathcal{B}(1/50) \times \overline{\mathbb{C}}$, *we have*

$$|H(y,z) - H(y',z')| \leq 2^{24}([z,z'] + d_{\mathcal{B}(1)}(y,y')).$$

PROOF. First we show the estimate

$$|\eta(z,w) - \eta(z',w')| \leq 2\left(\frac{1}{[z,w]} + \frac{1}{[z',w']}\right) \times ([z,z'] + [w,w']). \tag{8.22}$$

Indeed, since $\log x \leq x - 1$, we have

$$\eta(z,w) - \eta(z',w') = 2\log\frac{[z',w']}{[z,w]} \leq 2\frac{[z',w'] - [z,w]}{[z,w]}.$$

Using $[z',w'] \leq [z,w] + [z,z'] + [w,w']$, we obtain

$$\begin{aligned}
\eta(z,w) - \eta(z',w') &\leq 2\frac{[z,z'] - [w,w']}{[z,w]} \\
&\leq 2\left(\frac{1}{[z,w]} + \frac{1}{[z',w']}\right) \times ([z,z'] + [w,w']).
\end{aligned}$$

Similarly, we have

$$\eta(z',w')-\eta(z,w)\le 2\left(\frac{1}{[z,w]}+\frac{1}{[z',w']}\right)\times([z,z']+[w,w']).$$

Thus, we obtain (8.21).

We have

$$|H(y,z)-H(y',z')|\le\int_{\overline{\mathbb{C}}}|\eta(z,\hat\Phi(y,w))-\eta(z',\hat\Phi(y',w))|\omega_{\overline{\mathbb{C}}}[w]$$
$$+\int_{\overline{\mathbb{C}}}|\eta(\infty,\hat\Phi(y,w))-\eta(\infty,\hat\Phi(y',w))|\omega_{\overline{\mathbb{C}}}[w].$$

By (8.22), we have

$$|\eta(z,\hat\Phi(y,w))-\eta(z',\hat\Phi(y',w))|\le 2\left(\frac{1}{[z,\hat\Phi(y,w)]}+\frac{1}{[z',\hat\Phi(y',w)]}\right)$$
$$\times([z,z']+[\hat\Phi(y,w),\hat\Phi(y',w)]).$$

Since $\hat\Phi$ is holomorphic in $y\in\mathcal{B}(1)$, using (7.5), we have

$$\left[\hat\Phi(y,w),\hat\Phi(y',w)\right]\le 5d_{\mathcal{B}(1)}(y,y').$$

Hence, we obtain

$$|\eta(z,\hat\Phi(y,w))-\eta(z',\hat\Phi(y',w))|\le 10\left(\frac{1}{[z,\hat\Phi(y,w)]}+\frac{1}{[z',\hat\Phi(y',w)]}\right)$$
$$\times([z,z']+d_{\mathcal{B}(1)}(y,y')).$$

Thus, we have

$$|H(y,z)-H(y',z')|\le 10I([z,z']+d_{\mathcal{B}(1)}(y,y')), \tag{8.23}$$

where

$$I=\int_{\overline{\mathbb{C}}}\left(\frac{1}{[z,\hat\Phi(y,w)}+\frac{1}{[z',\hat\Phi(y',w)}+\frac{1}{[\infty,\hat\Phi(y,w)}+\frac{1}{[\infty,\hat\Phi(y',w)}\right)\omega_{\overline{\mathbb{C}}}[w].$$

By (8.21), we have $I<2^{20}$. Thus, by (8.23), we have

$$|H(y,z)-H(y',z')|\le 2^{24}([z,z']+d_{\mathcal{B}(1)}(y,y')).$$

□

Proof of Lemma 8.12. We first show (1). By Lemma 8.15, H is continuous on $\mathcal{B}(1/50)\times\overline{\mathbb{C}}$. Note that $p_2^*\omega_{\overline{\mathbb{C}}}-\Psi^*\omega_{\overline{\mathbb{C}}}$ is smooth on $\mathcal{B}(1)\times\overline{\mathbb{C}}$ outside $\Phi^{-1}(E)$. Hence it is enough to show (8.20) as currents of degree 2 on $\mathcal{B}(1/50)\times\overline{\mathbb{C}}$.

For $(y,z)\in\mathcal{B}(1/50)\times\overline{\mathbb{C}}$ and $w\in\overline{\mathbb{C}}$, we set

$$h_w(y,z)=\log(1+|z|^2)-\log|z-\hat\Phi(y,w)|^2.$$

Then we have

$$H(y, z) = \int_{\overline{\mathbb{C}}} h_w(y, z)\omega_{\overline{\mathbb{C}}}[w].$$

By the Poincaré-Lelong formula [26,p. 171], we have for each $w \in \mathbb{C}$,

$$dd^c h_w = \frac{1}{\pi} p_2^* \omega_{\overline{\mathbb{C}}} - \delta(z - \hat{\Phi}(y, w) = 0) \tag{8.24}$$

as $(1, 1)$-currents on $\mathcal{B}(1) \times \overline{\mathbb{C}}$.

Now let η be a test form. We have

$$\int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} H\, dd^c\eta = \int_{\overline{\mathbb{C}}} \left(\int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} h_w\, dd^c\eta \right) \omega_{\overline{\mathbb{C}}}[w],$$

$$\begin{aligned}\int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} \Psi^*\omega_{\overline{\mathbb{C}}} \wedge \eta &= \int_{w\in\overline{\mathbb{C}}} \left(\int_{\Phi^{-1}(w)} \eta \right) \omega_{\overline{\mathbb{C}}}[w] \\ &= \int_{w\in\overline{\mathbb{C}}} \left(\int_{(z - \hat{\Phi}(y, w) = 0)} \eta \right) \omega_{\overline{\mathbb{C}}}[w].\end{aligned}$$

Hence, by (8.24), we have

$$\begin{aligned}\int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} H\, dd^c\eta &= \int_{\overline{\mathbb{C}}} \left(\int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} \frac{1}{\pi} p_2^*\omega\overline{\mathbb{C}} \wedge \eta - \int_{(z - \hat{\Phi}(y, w) = 0)} \eta \right) \omega_{\overline{\mathbb{C}}}[w] \\ &= \int \mathcal{B}(1/50) \times \overline{\mathbb{C}} p_2^*\omega_{\overline{\mathbb{C}}} \wedge \eta - \int_{\mathcal{B}(1/50)\times\overline{\mathbb{C}}} \Psi^*\omega_{\overline{\mathbb{C}}} \wedge \eta.\end{aligned}$$

This shows (8.20) as currents. We complete the proof of Lemma 8.12(1).

Next we show Lemma 8.12(2). Let $\gamma = (\gamma_1, \gamma_2) : (-1, 1) \to \mathcal{B}(1/50) \times \overline{\mathbb{C}}$ be an arc such that $\gamma(0) = (y, z)$ and $\dot{\gamma} = -Jv$. Since

$$d^c H(v) = \frac{1}{4\pi} dH(-Jv),$$

we have

$$\begin{aligned}|d^c H(v)| &= \frac{1}{4\pi}|dH(-Jv)| \\ &= \frac{1}{4\pi}\left|\lim_{t\to 0} \frac{H(\gamma(t)) - H(\gamma(0))}{t}\right| \\ &\leq 2^{22}\left(\overline{\lim_{t\to 0}} \frac{d_{\mathcal{B}(1)(\gamma_1(t),y)+[\gamma_2(t),z]}}{|t|}\right) \quad \text{by Lemma 8.15} \\ &\leq 2^{22}(\|-Jv_1\|_{\mathcal{B}(1)+\|-Jv_2\|_{\overline{\mathbb{C}}}},\end{aligned} \tag{8.25}$$

where $\|\cdot\|_{\mathcal{B}(1)}$ is the infinitesimal Kobayashi metric on $\mathcal{B}(1)$. For the last estimate, see [21, p. 95, Lemma 3.5.33]. Using $\|-Jv_1\|_{\mathcal{B}(1)} = \|v_1\|_{\mathcal{B}(1)}$ and $\|-Jv_2\|_{\overline{\mathbb{C}}} = \|v_2\|_{\overline{\mathbb{C}}}$, we obtain

$$|d^c H(v)| \leq 2^{22}(\|v_1\|_{\mathcal{B}(1)} + \|v_2\|_{\overline{\mathbb{C}}}). \tag{8.26}$$

Next we show

$$\|v_1\|_{\mathcal{B}(1)} \leq 4\|v_1\|_{T(\overline{\mathbb{C}}\setminus E}. \tag{8.27}$$

To show this, we first claim that

$$\left\{w \in T(\overline{\mathbb{C}} \setminus E\,;\, d_{T(\overline{\mathbb{C}}\setminus E}([\mathrm{id}], w) < \log \sqrt{2}\right\} \subset \mathcal{B}(1), \tag{8.28}$$

where $d_{T(\overline{\mathbb{C}}\setminus E}$ is the Kobayashi-Teichmüller distance on $T(\overline{\mathbb{C}} \setminus E$. Indeed using (8.4), we have

$$d_{T(\overline{\mathbb{C}}\setminus E}([\mathrm{id}], w) \geq c_{\mathcal{B}(3)}(0, w) = d_\Delta(0, \|w\|_\infty/3).$$

Hence, if $d_{T(\overline{\mathbb{C}}\setminus E}([\mathrm{id}], w) < \log\sqrt{2} = d_\Delta(0, 1/3)$, then $\|w\|_\infty < 1$. Thus $w \in \mathcal{B}(1)$.

Now we set $t = \|v_1\|_{T(\overline{\mathbb{C}}\setminus E}$. There exists a holomorphic map $f : \Delta \to T(\overline{\mathbb{C}} \setminus E$ with

$$f(0) = y, \quad f_*\left(t\left(\frac{\partial}{\partial z}\right)_0\right) = v_1.$$

Since $d_\Delta(0, 47/149) = \frac{1}{2}\log\frac{98}{51}$, we have

$$f\left(\Delta\left(\frac{47}{149}\right)\right) \subset \left\{w \in T(\overline{\mathbb{C}} \setminus E\,;\, d_{T(\overline{\mathbb{C}}\setminus E}(y, w) < \frac{1}{2}\log\frac{98}{51}\right\}.$$

Since

$$d_{T(\overline{\mathbb{C}}\setminus E}([\mathrm{id}], y) \leq d_{\mathcal{B}(1)}(0, y) < d_\Delta(0, 1/50) = \frac{1}{2}\log\frac{51}{49},$$

we conclude from (8.28)

$$f\left(\Delta\left(\frac{47}{149}\right)\right) \subset \mathcal{B}(1).$$

Thus, we have

$$\|v_1\|_{\mathcal{B}(1)} \leq \|t\left(\frac{\partial}{\partial z}\right)_0\|_{\Delta(47/149)} = \frac{149}{47}t \leq 4\|v_1\|_{T(\overline{\mathbb{C}}\setminus E}.$$

This shows (8.27). Hence, by (8.26) and (8.27), we establish Lemma 8.12(2).

CHAPTER 9

Application of Ahlfors' theory of covering surfaces

9.1. Introduction

We have constructed a quasimeromorphic function g over the domain $\Omega \subset X(a_1, \ldots, a_q)$ which is described in Proposition 6.2. In this section, we apply Ahlfors' theory of covering surfaces to the quasimeromorphic function g to prove Proposition 7.2.

Base surface. Let $\{b_1, \ldots, b_q\} \subset \overline{\mathbb{C}}$ be a finite set of distinct points with $b_q = \infty$. Set $\Xi = \{b_1, \ldots, b_{q-1}\}$, which is a set of distinct points in

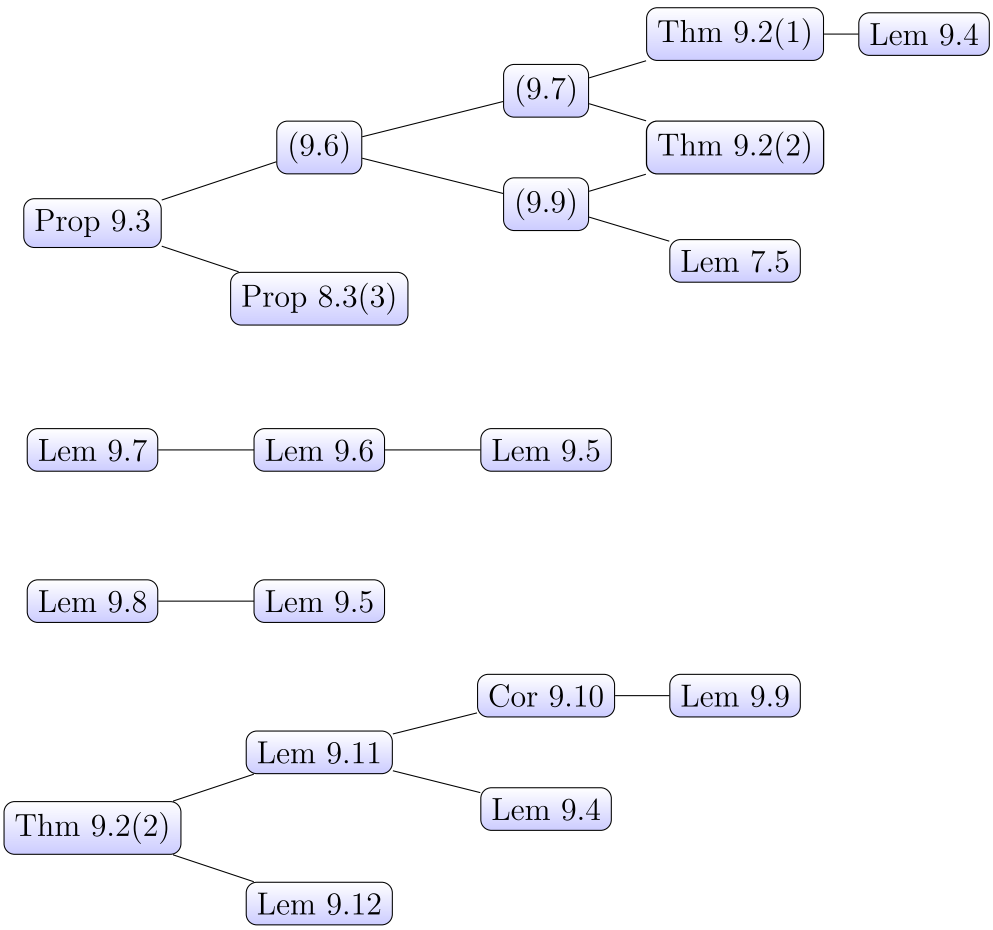


FIGURE 9.1. Structure of Chapter 9.

$\mathbb{C}$. For $s > 0$ and $i = 1, \ldots, q$, we define a disc $\Delta_i(s; b_1, \ldots, b_q)$ around the point b_i as follows; for $i = 1, \ldots, q-1$, we set

$$\Delta_i(s; b_1, \ldots, b_q) = \{z \in \mathbb{C} \,:\, |z - b_i| < s\varrho_i\},$$

where $\varrho_i = \min_{c \in \Xi \setminus \{b_i\}} |c - b_i|$. Thus for $s = 1$, the circle $\partial\Delta_i$ centered at b_i goes through the closest point b_j. If $s = 1/2$, the discs Δ_i are disjoint.

For $i = q$, we set

$$\Delta_q(s; b_1, \ldots, b_q) = \left\{ z \in \mathbb{C}; |z - b_{q-1}| > \frac{R}{s} \right\},$$

where $R = \max_{c \in \Xi \setminus \{b_{q-1}\}} |c - b_{q-1}|$. Given a constant $s < \frac{1}{10}$, we remove q disks $\Delta_i(s; b_1, \ldots, b_q)$ from the Riemann sphere to define the base surface

$$B(s; b_1, \ldots, b_q) = \overline{\mathbb{C}} - \bigcup_{i=1}^{q} \overline{\Delta_i(s; b_1, \ldots, b_q)}.$$

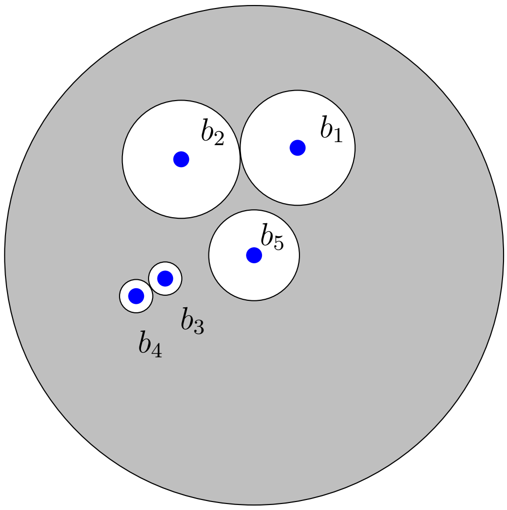


FIGURE 9.2. Base surface $B(1/2; b_1, \ldots, b_q)$.

Figure 9.2 contains an example in the case $q = 6$. Now $b_6 = \infty$. In this case, $s = 1/2$ and the discs $\Delta_i(s; b_1, \ldots, b_6)$, for $i = 1, \ldots, 6$ don't overlap.

The base surface B is contained in the complement of

$$\Delta_6(1/2; b_1, \ldots, b_6)$$

which is the disc

$$D(b_5, 2R), \quad R = \max_{c \in \Xi \setminus \{b_5\}} = |b_5 - b_2|.$$

The closest point to b_3 is b_4. Therefore

$$\rho_3 = \min_{c \in \Xi \setminus \{b_3\}} |c - b_3| = |b_4 - b_3|$$

and the disc

$$\Delta_3(1/2; b_1, \ldots, b_6) = D(b_3, \rho_3/2)$$

is deleted from B.

For an arc $\gamma \subset \overline{\mathbb{C}}$, we set

$$\ell_\Xi = \sum_{(b,c)\in\Xi\times\Xi\setminus\text{diagonal}} \ell_{\overline{\mathbb{C}}}(\varphi_{b,c}(\gamma)),$$

where the map $\varphi_{b,c}$ is defined by

$$\varphi_{b,c}(z) = \frac{z-b}{z-a}. \tag{9.1}$$

Example 9.1. If

$$\gamma = \{\gamma(t) = (1-t)b + tc\,;\, 0 \leq t \leq 1\},$$

then $\varphi_{b,c}(\gamma) = [0,1]$ and $\ell_{\overline{\mathbb{C}}}(\varphi_{b,c}(\gamma)) = \ell_{\overline{\mathbb{C}}}([0,1]) = \frac{\pi}{4}$.

For a subset $D \subset \overline{\mathbb{C}}$, we denote by $A(D)$ the area of D with respect to the spherical area form $\omega_{\overline{\mathbb{C}}}$, that is,

$$A(D) = \int_D \omega_{\overline{\mathbb{C}}}.$$

Notation from topology. If a domain D is bounded by a finite number of simple closed curves, we denote by $\varrho(D)$ the negative of the Euler characteristic of D. Since $B(s; b_1, \ldots, b_q)$ is a sphere with q-holes, we have

$$\varrho(B(s; b_1, \ldots, b_q)) = q - 2. \tag{9.2}$$

We set $\varrho^+(D) = \max\{\varrho(D), 0\}$.

We formulate the main results of Ahlfors' theory in the following form where the constants 'h' in [**12**] in the theory are controlled explicitely. The first statement should be compared with 'Covering theorem 1' [**12**, p. 328] and the second statement should be compared with 'Main theorem' [**12**, p. 332].

Theorem 9.2. *Assume that* $\{0,1\} \subset \Xi$ *and set* $B = B(s; b_1, \ldots, b_q)$, *where* $s < 1/10$ *and* $q \geq 3$.
(1) Let F be a finite covering surface of the Riemann sphere with a covering map $p : F \to \overline{\mathbb{C}}$. Then we have

$$\left| \frac{A(F)}{\pi} - \frac{A(p^{-1}(B))}{A(B)} \right| \leq \ell_{\overline{\mathbb{C}}}(\partial F). \tag{9.3}$$

(2) Assume that $(\overline{\mathbb{C}}, b_1, \ldots, b_q)$ is ε-thick, where $0 < \varepsilon < 1$. Let F be a finite covering surface of B with relative boundary $\partial' F$. Then we have

$$(q-2)\frac{A(F)}{A(B)} \leq \varrho^+(F) + \frac{2^{19}q^4}{\varepsilon}\ell_\Xi(\partial' F). \tag{9.4}$$

Here, area and length on a covering surface is measured using the pull back metric on the base surface.

To state the main result of this section, we need to introduce the following notation. Let D and G be two open sets in $\mathbb{C}$. We define two subsets $\mathcal{I}(D,G)$ and $\mathcal{P}(D,G)$ of the set of connected components of $D \cap G$ in the following manner. Let D' be a connected component of $D \cap G$, then D' is contained in $\mathcal{I}(D,G)$ if and only if D' is compactly contained in G, otherwise D' is contained in $\mathcal{P}(D,G)$.

PROPOSITION 9.3. *Let f be a transcendental meromorphic function in the complex plane and let $a_1, \dots, a_q \in \mathcal{R}_d$ be distinct with $a_q = \infty$, where $d \geq 1$ and $q \geq 3$. Assume that $x \in \Omega$ and $\Omega^* \Subset \Omega$ are the same as in Proposition 6.3. Assume $(\overline{\mathbb{C}}, a_1(x), \dots, a_q(x))$ is $1/2^{20}$-thick. Set $B = B(s; a_1(x), \dots, a_q(x))$, where $s < 1/10$. Set*

$$\chi(r,t) = \sum_{F \in \mathcal{I}(g^{-1}(B), \Omega(r,t)} \varrho(F) + \sum_{F \in \mathcal{P}(g^{-1}(B), \Omega(r,t)} \varrho^+(F),$$

where g is the quasiconformal perturbation of f defined by (8.1)*. Then for each distinct $i, j \in \{1, 2, \dots, q-1\}$, we have*

$$\begin{aligned} &(q-2)\int_0^{m/2} T\left(r, \frac{f-a_i}{a_j-a_i}, \Omega(t)\right) dt \\ &\leq \int_0^{m/2}\int_1^r \frac{\chi(u,t)}{u}\, du\, dt + \frac{2^{67} d q^8}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \end{aligned} \tag{9.5}$$

for $r > \gamma_d$.

9.2. Derivation of Proposition 9.3 from Theorem 9.2.

We set $\varphi = \varphi_{a_i(x), a_j(x)}$ to simplify the notation. We consider the quasimeromorphic function $\varphi \circ g$ on Ω. The main issue in our derivation is to derive

$$\begin{aligned} &(q-2)\int_0^{m/2} T(r, \varphi \circ g, \Omega(t)) dt \\ &\leq \int_0^{m/2}\int_1^r \frac{\chi(u,t)}{u}\, du\, dt + \frac{2^{66} d q^8}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \end{aligned} \tag{9.6}$$

for $r > \gamma_d$. Once (9.6) is established, Proposition 8.3(3) immediately implies (9.5). If g is constant, then (9.6) is obvious. Thus in the following, we assume that g is non-constant.

We first derive the following non-integrated version of (9.6) from Theorem 9.2:

$$(q-2)\frac{\int_{\Omega(r,t)} (\varphi \circ g)^* \omega_{\overline{\mathbb{C}}}}{\pi} \leq \chi(r,t) + 2^{40} q^4 \ell_{\Xi}(g(\partial\Omega(r,t)). \tag{9.7}$$

Let F be a connected component of $g^{-1}(B)\cap\Omega(r,t)$. We consider the restriction of $\varphi\circ g$ on F as a covering surface

$$\varphi\circ g|_F : F\to\varphi(B). \tag{9.8}$$

If F is compactly contained in $\Omega(r,t)$, that is, $F\in\mathcal{I}(g^{-1}(B),\Omega(r,t))$, then the covering (9.8) does not have a relative boundary. Hence, by the Hurwitz formula and (9.2), we have

$$(q-2)\frac{\int_F(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}}{A(\varphi(B))}\le\varrho(F).$$

Next we consider the case $F\in\mathcal{P}(g^{-1}(B),\Omega(r,t))$. Note that

$$\varphi(B)=B(s;\varphi(a_1(x)),\dots,\varphi(a_q(x)))$$

and

$$\{0,1\}\subset\{\varphi(a_1(x)),\dots,\varphi(a_q(x))\}.$$

Hence we may apply Theorem 9.2(2) to the covering (9.8), combined with (9.2), to obtain

$$(q-2)\frac{\int_F(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}}{A(\varphi(B))}\le\varrho^+(F)+2^{39}q^4\ell_\Xi(g(\partial\Omega(r,t)\cap\overline{F})).$$

Since

$$\sum_{F\in\mathcal{I}g^{-1}(B),\Omega(r,t)\cup\mathcal{P}(g^{-1}(B),\Omega(r,t))}\int_F(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}=\int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}},$$

we conclude

$$\begin{aligned}&(q-2)\frac{\int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}}{A(\varphi(B)}\\ &\le\sum_{F\in\mathcal{I}(g^{-1}(B),W)}\varrho(F)\\ &+\sum_{F\in\mathcal{P}(g^{-1}(B),w)}\varrho^+(F)+2^{39}q^4\ell_\Xi(g(\partial\Omega(r,t))).\end{aligned}$$

By Theorem 9.2(1), we have

$$\begin{aligned}&\frac{\int_{\Omega(r,t)(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}}}{\pi}\\ &\le\frac{\int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\overline{\mathbb{C}}}}{A(\varphi(B))}+\ell_{\overline{\mathbb{C}}}(\varphi\circ g(\partial\Omega(r,t))).\end{aligned}$$

Thus, we obtain (9.7).

Now taking the integral on both sides of (9.7), we obtain

$$(q-2)\int_0^{m/2} T(r,\varphi\circ g,\Omega(t))dt \le \int_0^{m/2}\frac{\chi(u,t)}{u}dudt + 2^{40}q^4\int_0^{m/2}\int_1^r \frac{\ell_\Xi(g(\partial\Omega(r,t))}{u}dudt.$$

We need to estimate the second term on the right-hand side. Let k and l be distinct elements from Ξ. We put $\varphi_{k,l}=\varphi_{a_k(x),a_l(x)}$ to simplify the notation[1]. We claim that

(9.9)

$$\int_0^{m/2}\int_1^r \frac{\ell_{\overline{\mathbb{C}}}(\varphi_{k,l}\circ g(\partial\Omega(u,t))}{u}\,du\,dt \le \frac{2^{26}dq^2}{m}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}$$

for $r>\gamma_d$. This estimate completes the derivation of (9.6), hence (9.5).

It remains to show (9.9). We set

$$\rho(z)=\frac{|(\varphi_{k,l}\circ g)_z(z)|+|(\varphi_{k,l}\circ g)_{\overline{z}}(z)}{1+|(\varphi_{k,l}\circ g)(z)|^2}.$$

Then we have

$$\ell_{\overline{\mathbb{C}}}(\varphi_{k,l}\circ g(\Omega(r,t)))\le\int_{\partial\Omega(r,t)}\rho(z)|dz|$$

$$\int_{\Omega(r,t)}\rho^2(z)|dz|^2\le K_g\int_{\Omega(r,t)}(\varphi_{k,l}\circ g)^*\omega_{\overline{\mathbb{C}}}.$$

Hence, by Proposition 8.3(3), we have for $r>\gamma_d$

$$\int_1^r\int_{\Omega(u,m/2)}\rho^2(z)|dz|^2\frac{du}{u}\le \pi K_g T(r,\varphi_{k,l}\circ g,\Omega(m/2))$$

$$\le \pi K_g T(r,\mathrm{cr}(f,a_k,a_l,a_q),\Omega(m))$$

$$+\frac{2^{30}dq^2\pi K_g}{m}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}$$

$$\le\frac{2^{32}dq^2}{m}T\left(r+\frac{1}{T(r)}\right).$$

Thus, we may apply Lemma 7.5 to

$$\Lambda(r)=\frac{2^{32}dq^2}{m}T\left(r+\frac{1}{2T(r)}\right),\quad \widetilde{\Lambda}(r)=\frac{2^{32}dq^2}{m}T\left(r+\frac{1}{T(r)}\right)$$

to obtain (9.9).

---

[1] recall that

$$\varphi_{b,c}(z)=\frac{z-b}{c-b}$$

9.3. Proof of Theorem 9.2(1)

First we recall isoperimetric inequalities on the sphere. Let γ be a simple closed curve on the Riemann sphere $\overline{\mathbb{C}}$. Then γ divides $\overline{\mathbb{C}}$ into two parts D_1 and D_2. The following inequalities are well known:

$$\min\{A(D_1), A(D_2)\} \leq \frac{1}{2}\ell_{\overline{\mathbb{C}}}(\gamma) \tag{9.10}$$

$$\min\{A(D_1), A(D_2)\} \leq \frac{1}{2\pi}\ell_{\overline{\mathbb{C}}}(\gamma)^2 \tag{9.11}$$

Equality holds if and only if γ is a great circle.

We start the proof of Theorem 9.2(1). We decompose F into the sheets $G_1, \dots, G_n$ as in [**12**, p. 323]. Namely G_j is the part of $\overline{\mathbb{C}}$ where the covering $p : F \to \overline{\mathbb{C}}$ has at least j preimages with counting multiplicities. Then we have

$$A(F) = \sum_{j=1}^{n} A(G_j),$$

$$\frac{A(p^{-1}(B))}{A(B)} = \sum_{j=1}^{n} \frac{A(G_j \cap B)}{A(B)}$$

$$\ell_{\overline{\mathbb{C}}}(\partial F) \geq \sum_{j=1}^{n} \ell_{\overline{\mathbb{C}}}(\partial G_j).$$

Hence, it is enough to show

$$\left|\frac{A(G_j)}{\pi} - \frac{A(G_j \cap B)}{A(B)}\right| \leq \ell_{\overline{\mathbb{C}}}(\partial G_j). \tag{9.12}$$

Let D_j be one of G_j or $\overline{\mathbb{C}} \setminus G_j$ which has smaller area. Then by (9.10), we have

$$A(D_j) \leq \frac{1}{2}\ell_{\overline{\mathbb{C}}}(\partial G_j).$$

Thus, by Lemma 9.4, we obtain

$$\left|\frac{A(D_j)}{\pi} - \frac{A(D_j \cap B)}{A(B)}\right| \leq 2A(D_j) \leq \ell_{\overline{\mathbb{C}}}(\partial G_j).$$

If $D_j = G_j$, this is what we want to show. If $D_j = \overline{\mathbb{C}} \setminus G_j$, by

$$\frac{A(G_j)}{\pi} = 1 - \frac{A(D_j)}{\pi}, \quad \frac{A(G_j \cap B)}{A(B)} = 1 - \frac{A(D_j \cap B)}{A(B)},$$

we obtain (9.12). Hence we conclude the proof of Theorem 9.2(1).

LEMMA 9.4. *Let Ξ and B be the same as in Theorem 9.2. Then we have*

$$A(B) > 1.$$

PROOF. We remark that $\Delta_i(\frac{1}{5}; b_1, \dots, b_q)$ is contained in some hemisphere for $i = 1, \dots, q$. This is immediate from the definition if $b_i = 0$ or $i = q$. For the other discs, this follows from the fact that $\Delta_i(\frac{1}{5}; b_1, \dots, b_q) \cap \{0, \infty\} = \emptyset$.

We consider the annulus

$$R_i = \Delta_i(\frac{1}{5}; b_1, \dots, b_q) \setminus \overline{\Delta_i(s; b_1, \dots, b_q)}.$$

Then, for the modulus of R_i, we have

$$\mathrm{Mod}(R_i) \geq \frac{\log 2}{2\pi} > \frac{1}{4\pi}.$$

Let Γ be the set of all closed curves in R_i which separate the two boundary circles of R_i. Then we have

$$\frac{\inf_{\gamma \in \Gamma} \ell_{\overline{\mathbb{C}}}(\gamma)^2}{A(R_i)} \leq \frac{1}{\mathrm{Mod}(R_i)}.$$

Hence, by (9.11), we have

$$A(R_i) \geq \frac{1}{4\pi} \inf_{\gamma \in \Gamma} \ell_{\overline{\mathbb{C}}}(\gamma)^2 \geq \frac{1}{2} A(\Delta_i(s; b_1, \dots, b_q)).$$

Since

$$A(R_i) = A(\Delta_i(\frac{1}{5}; b_1, \dots, b_q)) \setminus A(\Delta_i(s; b_1, \dots, b_q)),$$

we have

$$\frac{2}{3} A(\Delta_i(\frac{1}{5}; b_1, \dots, b_q)) \geq A(\Delta_i(s; b_1, \dots, b_q)).$$

Since the discs $\Delta_i(\frac{1}{5}; b_1, \dots, b_q)$, $1 \leq i \leq q$, are disjoint, we have

$$\begin{aligned} A(B) &= A(\overline{\mathbb{C}}) - \sum_{i=1}^{q} A(\Delta_i(s; b_1, \dots, b_q)) \\ &\geq A(\overline{\mathbb{C}}) - \frac{2}{3} \sum_{i=1}^{q} A(\Delta_i(\frac{1}{5}; b_1, \dots, b_q)) \\ &\geq \frac{1}{3} A(\overline{\mathbb{C}}) = \frac{\pi}{3} > 1. \end{aligned}$$

This proves our lemma. □

9.4. Minimal spanning tree

We recall the set $\Xi = \{b_1, \dots, b_{q-1}\}$. We denote by $\Gamma_{\min}$ the minimal spanning tree for Ξ. By definition, a spanning tree Γ is a collection of line segments with end points in Ξ such that Γ contains a path connecting every pair of points $b_i, b_j \in \Xi$, and such that Γ contains no closed path. The minimal spanning tree is a spanning tree for which the total Euclidean length of line segments is minimal. We collect elementary properties of $\Gamma_{\min}$.

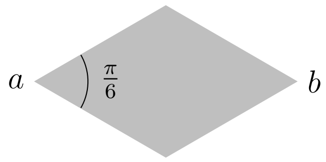


FIGURE 9.3. Set $K_{\overline{ab}}$

LEMMA 9.5. *Let $c_1, \ldots, c_n \in \Xi$ be distinct points. Then one of the longest setments in $\overline{c_1c_2}$, $\overline{c_2c_3}$,..., $\overline{c_{n-1}c_n}$, $\overline{c_nc_1}$ is not contained in $\Gamma_{\min}$.*

PROOF. Assume contrary that all longest segments are contained in $\Gamma_{\min}$. We may assume without loss of generality that $\overline{c_nc_1}$ is a longest segment. Then $\overline{c_nc_1}$ is contained in $\Gamma_{\min}$. We remove the segment $\overline{c_nc_1}$ from $\Gamma_{\min}$. The resulting graph consists of two connected components Γ and Γ': one, say Γ contains c_1 and the other, say Γ', contains c_n. Now let i, $1 \leq i \leq n$, be the largest number such that c_i is contained in Γ. Then c_{i+1} is contained in Γ'. Hence the segment $\overline{c_ic_{i+1}}$ is not contained in $\Gamma_{\min}$. Thus, we have $\overline{c_ic_{i+1}} < \overline{c_nc_1}$.

Now we add the line segment $\overline{c_ic_{i+1}}$ to $\Gamma \cup \Gamma'$. Then we obtain a new spanning tree for Ξ. Since $\overline{c_ic_{i+1}} < \overline{c_nc_1}$, the total length of this new spanning tree is strictly smaller than $\Gamma_{\min}$. This is a contradiction. Thus we have proved our lemma. □

LEMMA 9.6. *Let $a, b, c, d \in \Xi$ be distinct four points such that the line segments $\overline{ab}$ and $\overline{cd}$ are contained in $\Gamma_{\min}$. Then $\overline{ab}$ and $\overline{cd}$ do not intersect.*

PROOF. Assume contrary that $\overline{ab}$ and $\overline{cd}$ intersect. Then at least one of the four angles $\angle acb$, $\angle cbd$, $\angle bda$ and $\angle dac$ is greater than or equal to $\pi/2$. We may assume that $\angle acb \geq \pi/2$. Then we have $\overline{ac} < \overline{ab}$ and $\overline{cb} < \overline{ab}$. By Lemma 9.5, the segment $\overline{ab}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma. □

For a line segment $\overline{ab}$ contained in $\Gamma_{\min}$, we set

$$K_{\overline{ab}} = \{z \in \mathbb{C} \setminus \{a, b\} \; ; \angle zab < \pi/6 \quad \text{and} \quad \angle zba < \pi/6\}.$$

LEMMA 9.7. *Let $\overline{cd}$ be a line segment contained in $\Gamma_{\min}$ which is different from $\overline{ab}$. Then $K_{\overline{ab}}$ does not intersect with $\overline{cd}$.*

PROOF. We prove our lemma in two cases.

Case 1: $\# \{a, b, c, d\} = 3$. In this case, we may assume without loss of generality that $a = c$. For the sake of contradiction, we assume that the segment $\overline{cd}$ intersects with $K_{\overline{ab}}$. Then, since $\angle bcd < \pi/6$, we have $\overline{bd} < \max\left\{\overline{cb}, \overline{cd}\right\}$. Thus, by Lemma 9.5, the longer segment of $\overline{cb}$ and $\overline{cd}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma in the case $\# \{a, b, c, d\} = 3$.

Case 2: $\#\{a,b,c,d\} = 4$. We first prove $c, d \in \overline{K_{\overline{ab}}}$. Indeed, if $c \in \overline{K_{\overline{ab}}}$, then we have $\overline{ac} < \overline{ab}$ and $\overline{bc} < \overline{ab}$. Hence by Lemma 9.5, the segment $\overline{ab}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, □

LEMMA 9.8. *Let $b_i, b_j, b_k \in \Xi$ be distinct. Assume that the segment $\overline{b_j b_k}$ is contained in $\Gamma_{\min}$. Then $\Delta_i(1/\sqrt{2}; b_1, \dots, b_q)$ does not intersect with $\overline{b_j b_k}$.*

PROOF. Assume contrary that $\Delta_i(1/\sqrt{2}; b_1, \dots, b_q)$ intersects with $\overline{b_j b_k}$. Then since $\overline{b_i b_j} \geq \varrho_i$ and $\overline{b_i b_k} \geq \varrho_i$, we have $\angle b_j b_i b_k > \pi/2$. Hence, we have $\overline{b_i b_j} < \overline{b_j b_k}$ and $\overline{b_i b_k} < \overline{b_j b_k}$. Thus by Lemma 9.5, the segment $\overline{b_j b_k}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma. □

9.5. Ahlfors regularity

We recall $B = B(s; b_1, \dots, b_q)$. Let $\alpha_1, \dots, \alpha_{q-2}$ be the line segments of $B \cap \Gamma_{\min}$. Take $b \in \Xi$ such that

$$|b - b_{q-1}| = \max_{c \in \Xi \setminus \{b_{q-1}\}} |c - b_{q-1}|,$$

and set

$$\alpha_{q-1} = B \cap \{z; z = b + t(b - b_{q-1}), t \geq 0\}. \tag{9.13}$$

We cut B by these line segments $\alpha_1, \dots, \alpha_{q-1}$ to obtain a simply connected bordered surface B'. Then $\partial B'$ contains the line segments

$$\beta_1, \beta_1', \dots, \beta_{q-1}, \beta_{q-1}',$$

where β_i and β_i' are two copies of α_i. We have

$$\partial B' = \beta_1 \cup \beta_1' \cup \cdots \cup \beta_{q-1} \cup \beta_{q-1}' \cup \partial B.$$

LEMMA 9.9 (Ahlfors regularity). *Assume that $(\overline{\mathbb{C}}, b_1, \dots, b_q)$ is ε-thick, where $0 < \varepsilon < 1$. Let γ be a cross cut of B', which divides $\partial B'$ into two parts σ_1 and σ_2. Then we have*

$$\min\{\ell_\Xi(\sigma_1), \ell_\Xi(\sigma_2)\} \leq \frac{2^{15} q^3}{\varepsilon} \ell_\Xi(\gamma). \tag{9.14}$$

PROOF. For all distinct $b, c \in \Xi$, we have

$$\ell_{\overline{\mathbb{C}}}(\varphi_{b,c}(\partial B)) \leq q\pi; \quad \ell_{\overline{\mathbb{C}}}(\varphi_{b,c}(\alpha_i)) \leq \pi.$$

Hence, we obtain

$$\ell_\Xi(\partial B') \leq 3\pi q^3. \tag{9.15}$$

Let $P, Q \in \partial B'$ be the end points of γ. We set $\Delta_i(r) = \Delta_i(r; b_1, \dots, b_q)$. We prove (9.14) in two cases whether one of P or Q is contained in $\Delta_i(1/10)$ for some $i = 1, \dots, q$ or not.

Case 1: One of P or Q is contained in $\Delta_i(1/10)$ for some $i = 1,\dots,q$. In this case, we assume that $P \in \Delta_i(1/10)$. The proof is divided in two cases.

Subcase 1-1: $i \neq q$. Let $b_j \in \Xi$ be a point with

$$|b_j - b_i| = \min_{c\in\Xi\setminus\{b_i\}} |c - b_i|. \tag{9.16}$$

If γ is not contained in $\Delta_i(1/2)$, then a subarc of γ connects $\partial\Delta_i(1/10)$ and $\partial\Delta_i(1/2)$. Hence, we have

$$\ell_{\text{Euclid}}(\varphi_{i,j}(\gamma)) \geq \frac{2}{5},$$

where[2]

$\varphi_{i,j} = \varphi_{b_i,b_j}$. In general, for an arc γ' contained in the disc $\left\{|z| < \frac{1}{2}\right\}$, we have

$$\ell_{\overline{\mathbb{C}}}(\gamma') \geq \frac{4}{5}\ell_{\text{Euclid}}(\gamma'). \tag{9.17}$$

Hence, we have

$$\ell_{\Xi}(\gamma) > \ell_{\overline{\mathbb{C}}}(\varphi_{i,j}(\gamma)) > \frac{2}{5} \times \frac{4}{5} = \frac{8}{25}.$$

Thus, by (9.15), we obtain the estimate (9.14).

Next we assume that γ is contained in $\Delta_i(1/2)$. Let σ_1 be the part of $\partial B'$ which is contained in $\Delta_i(1/2)$. Then by Euclidean geometry and Lemmas 9.7 and 9.8, we have

$$\ell_{\text{Euclid}}(\sigma_1) < 6\ell_{\text{Euclid}}(\gamma).$$

Hence, by (9.17), we have

$$\ell_{\overline{\mathbb{C}}}(\varphi_{i,j}(\sigma_1)) \leq \frac{15}{2}\ell_{\overline{\mathbb{C}}}(\varphi_{i,j}(\gamma)). \tag{9.18}$$

Claim 1. Let $b_s, b_t \in \Xi$ be distinct. It is enough to show the estimate

$$X = \frac{|\varphi'_{s,t}(z)|}{1+|\varphi_{s,t}(z)|^2}\frac{1+|\varphi_{i,j}(z)|^2}{|\varphi'_{i,j}(z)|} \leq \frac{5}{4}$$

for $z \in \Delta_i(\frac{1}{2})$.

Denote

$$|\varphi_{s,t}(z)|^2 = \frac{A^2}{B^2}, \quad |\varphi'_{s,t}(z)| = \frac{1}{B}, \quad |\varphi_{i,j}(z)|^2 = \frac{C}{D}, \quad |\varphi'_{i,j}(z)| = \frac{1}{D}.$$

[2]We have the function

$$\varphi_{s,t}(z) = \varphi_{b_s,b_t}(z) = \frac{z-b_s}{b_t-b_s}, \quad \varphi'_{s,t}(z) = \frac{1}{b_t-b_s}.$$

which was defined on (7.1) at [**19**, page 757]
and abbreviated at[**19**, page 760, line 5].

Now

$$X = \frac{1}{B}\frac{B^2}{B^2+A^2}\frac{D^2+C^2}{D^2}\frac{D}{1} = \frac{B}{D}\frac{D^2+C^2}{B^2+A^2}.$$

Case (i) $s = i$. In this case, we have $|b_j - b_i| \le |b_t - b_s|$, means $D \le B$. Hence for $|z - b_i| \le |b_j - b_i|/2$, means $C \le D/2$, we have

$$X = \frac{B}{D}\frac{D^2+C^2}{B^2+A^2} \le \frac{B}{D}\frac{D^2+D^2/4}{B^2} = \frac{5}{4}\frac{D}{B} \le \frac{5}{4}.$$

Case (ii) $s \ne i$. In this case, we have $|z - b_s| \ge |b_j - b_i|/2$, means $A \ge D/2$. Hence for $|z - b_i| \le |b_j - b_i|/2$, means $C \le D/2$, we have

$$X = \frac{B}{D}\frac{D^2+C^2}{B^2+A^2} \le \frac{1}{D}\frac{D^2+D^2/4}{B+A^2/B} = \frac{5}{4}\frac{D}{2A} \le \frac{5}{4}.$$

Here we used the fact that

$$\frac{1}{B+A^2/B} \le \frac{1}{2A},$$

which comes from

$$h(x) = \frac{1}{x+a^2/x} = \frac{x}{x^2+a^2}, \quad h'(x) = \frac{a^2-x^2}{(x^2+a^2)^2}.$$

This proves our claim. □

Now by (9.18) and claim above, we have

$$\ell_\Xi(\sigma_1) < \frac{5q^2}{4}\ell_{\overline{\mathbb{C}}}(\varphi_{i,j}(\sigma)) \le \frac{75q^2}{8}\ell_\Xi(\gamma).$$

This shows our estimate (9.14).

Subcase 1-2: $i = q$. Let $b_j \in \Xi$ be a point with

$$|b_j - b_{q-1}| = \max_{c \in \Xi \setminus \{b_{q-1}\}} |c - b_{q-1}|. \tag{9.19}$$

If γ is not contained in $\Delta_q(1/2)$, then as in Subcase 1-1, we have

$$\ell_{\overline{\mathbb{C}}}(\varphi_{q-1,j}(\gamma)) > \frac{2}{5} \times \frac{4}{5} = \frac{8}{25}.$$

Thus, by (7.13), we obtain the estimate (7.12).

Next we assume that γ is contained in $\Delta_q(1/2)$. Let σ_1 be the part of $\partial B'$ which is contained in $\Delta_q(1/2)$. Then as in Subcase 1-1, we have

$$\ell_{\overline{\mathbb{C}}}(\varphi_{q-1,j}(\sigma_1)) \le \frac{15}{2}\ell_{\overline{\mathbb{C}}}(\varphi_{q-1,j}(\gamma)) \tag{9.20}$$

Claim 2.

$$\ell_\Xi(\sigma_1) \le 10q^2\ell_{\overline{\mathbb{C}}}(\varphi_{q-1,j}(\gamma)).$$

Proof of Claim 2. Let $b_s, b_t \in \Xi$ be distinct. It is enough to show the estimate

$$A = \frac{|\varphi'_{s,t}(z)|}{1+|\varphi_{s,t}(z)|^2}\frac{1+|\varphi_{q-1,j}(z)|^2}{|\varphi'_{q-1,j}(z)|} \le 10$$

for $z \in \Delta_q(1/2)$. We have

$$A = \frac{|b_t - b_s|}{|b_j - b_{q-1}|} \times \frac{|b_j - b_{q-1}|^2 + |z - b_{q-1}|^2}{|b_t - b_s|^2 + |z - b_s|^2}.$$

By (9.19),

$$\frac{|b_t - b_s|}{|b_j - b_{q-1}|} \leq \frac{|b_t - b_{q-1}| + |b_s - b_{q-1}|}{|b_j - b_{q-1}|} \leq \frac{2|b_j - b_{q-1}|}{|b_j - b_{q-1}|} = 2.$$

We can use the parallellogram identity

$$|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2, \quad x, y \in \mathbb{C},$$

with $x = z - b_s$ and $y = b_s - b_{q-1}$ to obtain

$$|z - b_{q-1}|^2 \leq 2|b_s - b_{q-1}|^2 + 2|z - b_s|^2.$$

Hence

$$\begin{aligned} A &\leq 2\frac{|b_j - b_{q-1}|^2 + |z - b_{q-1}|^2}{|z - b_s|^2} \\ &\leq 2\frac{|b_j - b_{q-1}|^2 + 2|b_s - b_{q-1}|^2 + 2|z - b_s|^2}{|z - b_s|^2} \\ &\leq 4 + \frac{2|b_j - b_{q-1}|^2 + 4|b_s - b_{q-1}|^2}{|z - b_s|^2}. \end{aligned}$$

Since $z \in \Delta_q(1/2)$ (see definition of $\Delta_q(1/2)$ from page 757), we have

$$|z - b_{q-1}| > \frac{R}{s} = 2R \geq 2|c - b_{q-1}|, \quad c \in \Xi.$$

Choose $c = b_s$ to obtain

$$|z - b_{q-1}| \geq 2|b_s - b_{q-1}|.$$

Hence,

$$\begin{aligned} |z - b_s| &\geq |z - b_{q-1}| - |b_s - b_{q-1}| \\ &\geq 2|b_s - b_{q-1}| - |b_s - b_{q-1}| = |b_s - b_{q-1}|. \end{aligned} \tag{9.21}$$

Also, $b_j \in \Xi$ satisfies by (9.19)

$$|b_j - b_{q-1}| \geq |c - b_{q-1}| = |b_s - b_{q-1}|.$$

Hence

$$A \leq 4 + \frac{6|b_j - b_{q-1}|^2}{|z - b_s|^2} \leq 4 + 6 = 10.$$

Hence, by (9.20) and the claim above, we have

$$\ell_\Xi(\sigma_1) \leq 10q^2 \ell_{\overline{\mathbb{C}}}(\varphi_{q-1,j}(\sigma_1)) \leq 75q^2 \ell_\Xi(\gamma).$$

This shows our estimate (9.14).

Case 2: Both P and Q lie outside of $\bigcup_{i=1}^q \Delta_i(1/10)$. Since $\partial B \subset \bigcup_{i=1}^q \Delta_i(1/10)$, P and Q are contained in the line segments $\beta_1, \beta_1', \ldots, \beta_{q-1}, \beta_{q-1}'$.

Subcase 2-1: Both P and Q are contained in one of the same line segment in $\beta_1, \beta_1', \ldots, \beta_{q-1}, \beta_{q-1}'$. In this case, we may assume that P and Q are contained in β. We first observe that for all $c \in \Xi$,

$$\angle PcQ < \frac{\pi}{2}. \tag{9.22}$$

This is obvious if $i = q-1$. We consider the case $i \neq q-1$. Let $\overline{ab} \in \Gamma_{\min}$ bhe the line segment containing α_i, where $a, b \in \Xi$. The estimate (9.22) is obviously true if c is equal to a or b. Let $c \in \Xi$ be different from a and b. Then, by Lemma (9.5), we have

$$\angle PcQ < \angle acb < \frac{\pi}{2}.$$

Thus, we have proved (9.22).

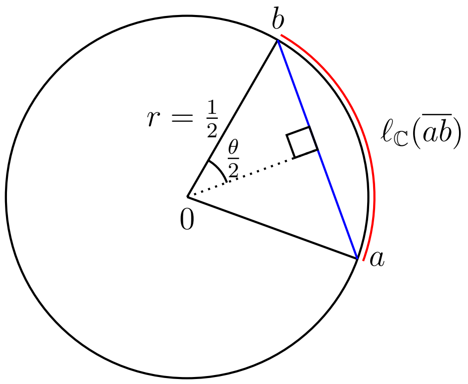
By (9.22), we have

$$\angle \varphi_{c,d}(P) 0 \varphi_{c,d}(Q) < \frac{\pi}{2}$$

for all distinct $c, d \in \Xi$.

Claim 3. Let $\zeta, \eta \in \mathbb{C} \setminus \{0\}$ be distinct. Then we have

$$[\zeta, \eta] < \ell_{\mathbb{C}}(\overline{\zeta\eta}) < \frac{\pi}{2}[\zeta, \eta]. \tag{9.23}$$



Geometric proof of Claim 3. Let $\theta = \angle \zeta 0 \eta < \pi/2$. By elementary trigonometry, $[\zeta, \eta] = \sin(\theta/2)$, while $\ell_{\mathbb{C}}(\zeta, \eta) = \theta/2$. We see that

$$\sin x \geq \frac{2}{\pi} x, \quad 0 \leq x \leq \pi.$$

Namely,

$$h(x) = \sin(x) - \frac{2}{\pi} x$$

satisfies $h(0) = 0$, $h(\pi/2) = 0$ and $h'(0) > 0$. We obtain that

$$\frac{\ell_{\mathbb{C}}(\zeta, \eta)}{[\zeta, \eta]} = \frac{\theta}{2\sin(\theta/2)} \leq \frac{\theta}{2\frac{2}{\pi}\frac{\theta}{2}} = \frac{\pi}{2},$$

as desired. $\square$

.

See the exercises or Yamanoi's paper for an analytic proof.

Subcase 2-2: P and Q are contained in two different line segments in $\beta_1, \beta_1', \dots, \beta_{q-1}, \beta_{q-1}'$. We assume that P is contained in β_i. In this case, we shall prove

$$\ell_\Xi(\gamma) > \frac{\varepsilon}{2440}. \tag{9.24}$$

First we consider the case $i = q-1$. Let $b \in \Xi$ be the point which appears in (9.13) and set

$$K = \{z \in \mathbb{C} \setminus \{b\}\,;\angle zbP < \pi/6\}.$$

Since $K \cap \Gamma_{\min} = \emptyset$, we have

$$\gamma \not\subset K. \tag{9.25}$$

Claim 4. There exists $c \in \Xi \setminus \{b\}$ such that

$$\frac{\varepsilon}{10} \le |\varphi_{b,c}(P)| \le 10. \tag{9.26}$$

Proof of Claim 4. We order $\Xi \setminus \{b\} = \{c_1, \dots, c_{q-2}\}$ such that

$$|c_1 - b| \le \cdots \le |c_{q-2} - b|.$$

By the assumption made in Case 2, we have

$$\frac{|c_1 - b|}{10} \le |P - b| \le 10|c_{q-2} - b|.$$

Case (i) $|P-b| \le |c_1 - b|$. In this case, we set $c = c_1$.

Case (ii) $P - b| \ge |c_{q-2} - b|$. In this case, we set $c = c_{q-2}$.

Case (iii) $|c_1 - b| < |P - b| < |c_{q-2} - b|$. In this case, we take j such that $|c_{j-1} - b| \le |P - b| \le |c_j - b|$. Since $(\overline{\mathbb{C}}, b_1, \dots, b-q)$ is ε-thick, we have

$$\frac{|c_{j-1} - b|}{|c_j - b|} \ge \varepsilon.$$

We set $c = c_j$. Then we have $\varepsilon \le |\varphi_{b,c}(P)| \le 1$. Thus, we have proved Claim 4. □

Now by Claim 4, we may take c such that (9.26) holds. Then by (9.25), the arc γ intersects with ∂K. Hence, we have

$$\ell_{\mathrm{Euclid}}(\varphi_{b,c}(\gamma)) \ge \frac{\varepsilon}{20}.$$

In general, for an arc γ' contained in the disc $\{|z| < 11\}$, we have

$$\ell_{\overline{\mathbb{C}}}(\gamma') \ge \frac{1}{122}\ell_{\mathrm{Euclid}}(\gamma').$$

Hence, we obtain (9.24).

Next we consider the case $i \ne q-1$. Let $\overline{ab}$ be the line segment which contains α_i. Then, by Lemma 9.7, $\gamma \not\subset K_{\overline{ab}}$. Let $w \in \partial K_{\overline{ab}}$ be the first point where γ and $\partial K_{\overline{ab}}$ intersect. We may assume without loss of generality that $\angle wbP = \pi/6$. By Claim 4, we may take c such that (9.26) holds. Here, we remark that Claim 4 is proved for the case $i = q-1$, but the proof shows that the same statement is valid for $i \neq q-1$. By the same argument as in the previous case, we obtain (9.24).

Now by (9.15) and (9.24), we conclude the proof of Lemma 9.9. We note $7320\pi < 2^{15}$. □

COROLLARY 9.10. *Assume that* $(\overline{\mathbb{C}}, b_1, \dots, b_q)$ *is* ε*-thick, where* $0 < \varepsilon < 1$. *Let* γ *be a loop cut or a cross cut of* B'*, which divides* B' *into two parts. Then one of them, denoted by* D*, satisfies the following two estimates:*

$$\ell_{\Xi}(\overline{D} \cup \partial B') < \frac{2^{15} q^3}{\varepsilon} \ell_{\Xi}(\gamma)$$
$$A(D) < \frac{2^{17} q^3}{\varepsilon} \ell_{\Xi}(\gamma).$$

PROOF. We first remark that every Jordan domain $D \subset \overline{B}$ satisfies

$$A(D) \leq 2\ell_{\overline{\mathbb{C}}}(\partial D). \tag{9.27}$$

When $\ell_{\overline{\mathbb{C}}}(\partial D) \geq \pi/2$, this is obvious. If $\ell_{\overline{\mathbb{C}}}(\partial D) < \pi/2$, then ∂D is contained in some hemisphere. Hence $D \subset \overline{\mathbb{C}}$ is contained in some hemisphere; otherwise $\overline{\mathbb{C}} \setminus D$ should be contained in some hemisphere, which is impossible by $0, 1, \infty \in \overline{\mathbb{C}} \setminus D$. Hence (9.27) is true.

Now if γ is a loop cut, then we take D such that $D \Subset B'$. Then by (9.27), the second estimate holds. The first one is trivial.

If γ is a cross cut, then we take D such that $\ell_{\Xi}(\overline{D} \cap \partial B')$ is shorter. Then by Lemma 9.9, the first estimate holds. Now D is bounded by the closed curve $\gamma \cup (\overline{D} \cap \partial B')$. Hence by (9.27), we obtain the second estimate. □

9.6. Proof of Theorem 9.2(2)

First we prove 'Covering theorem 2' [**12**, p. 329] in our particular situation. In the following, area and length are always measured using $\omega_{\overline{\mathbb{C}}}$ and ℓ_{Ξ}, respectively.

LEMMA 9.11. *Assume that* $(\overline{\mathbb{C}}, b_1, \dots, b_q)$ *is* ε*-thick, where* $0 < \varepsilon < 1$. *Let* G *be a covering surface of* B'. *Let* S *be the mean sheet number and* L *be the length of the relative boundary. For a line segment* $\beta \in \{\beta_1, \beta_1', \dots, \beta_{q-1}, \beta_{q-1}'\}$ *in* $\partial B'$*, let* $S(\beta)$ *be the mean sheet number over* β*. Then we have*

$$|S - S(\beta)| \leq \frac{2^{18} q^3}{\varepsilon} L.$$

PROOF. We decompose G into sheets $G_1, \dots, G_n$ as in [**12**, p. 323]. Thus, $G_j \subset B'$ is the part where the covering G has at least j preimages. Let S_j be the mean sheet number, $S_j(\beta)$ be the mean sheet number over β, and L_j be the length of the relative boundary of the jth sheet

G_j. Then we have

$$S = \sum_{j=1}^{n} S_j, \quad S(\beta) = \sum_{j=1}^{n} S_j(\beta), \quad L \geq \sum_{j=1}^{n} L_j.$$

We shall show

$$|S_j - S_j(\beta)| \leq \frac{2^{18}q^3}{\varepsilon} L_j. \tag{9.28}$$

for all $j = 1, \ldots, n$, which will establish our lemma.

We apply Corollary 9.10. Since each G_j is divided by loop cuts and cross cuts of total length L_j, one of G_j or $B' \setminus G_j$, which we write D_j, satisfies the two estimates of Corollary 9.10. Hence, by Lemma 9.4 and $\ell_\Xi(\beta) > 1$, we obtain

$$\left| \frac{A(G_j)}{A(B')} - \frac{\ell_\Xi(\overline{D} \cap \beta)}{\ell_\Xi(\beta)} \right| \leq \frac{2^{18}q^3}{\varepsilon} L_j. \tag{9.29}$$

If $D_j = G_j$, this is what we need to prove.

If $D_j = B' - G_j$, then we have

$$\frac{A(G_j)}{A(B')} = 1 - \frac{A(D_j)}{A(B')},$$
$$\frac{\ell_\Xi(\overline{G_j} \cap \beta)}{\ell_\Xi(B')} = 1 - \frac{\ell_\Xi(\overline{D_j} \cap \beta)}{\ell_\Xi(\beta)}.$$

Thus, by (9.29), we obtain the estimate (9.28). □

LEMMA 9.12. *Assume that $(\overline{\mathbb{C}}, b_1, \ldots, b_q)$ is ε-thick, where $0 < \varepsilon < 1$. Let F be a covering surface of B. Let S be the mean sheet number and let L be the length of the relative boundary. For a line segment α from $\{\alpha_1, \ldots, \alpha_{q-1}\}$, let $S(\alpha)$ be the mean sheet number over α with respect to ℓ_Ξ. Then we have*

$$|S - S(\alpha)| \leq \frac{2^{18}q^3}{\varepsilon} L.$$

PROOF. By deforming F slightly, if necessary, so that S, $S(\alpha)$, L change arbitrarily small, we may assume without loss of generality that the relative boundary of F has no arcs of positive length above $\alpha_1, \ldots, \alpha_{q-1}$ and that F has no branch cuts points above $\alpha_1, \ldots, \alpha_{q-1}$. Let $\{\sigma_j\}_{j=1}^{m}$ be the cross cuts of F over $\alpha_1, \ldots, \alpha_{q-1}$. By these cross cuts, F is divided in $G_1, \ldots, G_k$. Then each G_i is a covering surface of B'. Let S_i be the mean sheet number and let L_i be the length of the relative boundary of this covering. Then we have

$$S = \sum_{i=1}^{k} S_i, \quad L = \sum_{i=1}^{k} L_i.$$

Let β and β' be the line segments in $\partial B'$ that are two copies of α. Let $S_i(\beta)$ and $S_i(\beta')$ be defined as in Lemma 9.11. Then we have

$$|2S_i - S_i(\beta) - S_i(\beta')| \leq \frac{2^{19}q^3}{\varepsilon}L,$$

hence

$$\left|2S - \sum_{i=1}^{k}(S_i(\beta) + S_i(\beta'))\right| \leq \frac{2^{19}q^3}{\varepsilon}L.$$

Now each σ_j is contained in $\{\partial G_i\}_{i=1}^{k}$ exactly two times. Hence, we have

$$\sum_{i=1}^{k}(S_i(\beta) + S_i(\beta')) = 2S(\alpha).$$

This concludes the proof. □

Now we prove Theorem 9.2(2). We follow the proof of Toki [29], who simplified the original proof of Ahlfors [1]. As in the proof of Lemma 9.12, we may assume without loss of generality that the relative boundary of F has no arcs of positive length above $\alpha_1, \ldots, \alpha_{q-1}$ and that F has no branch points above $\alpha_1, \ldots, \alpha q - 1$. Let $\{\sigma_j\}_{j=1}^{m}$ be the cross cuts of F over $\alpha_1, \ldots, \alpha_{q-1}$. Given σ_j, which lies over α_k, we set

$$\lambda(\sigma_j) = \frac{\ell_\Xi(\sigma_j)}{\ell_\Xi(\alpha_k)}.$$

We have

$$0 \leq \lambda(\sigma_j) \leq 1. \tag{9.30}$$

We consider two cases.

Case 1: There exists σ_j which does not divide F. In this case, we may assume that $\sigma_1, \ldots, \sigma_n$ satisfies

(1) $F \setminus (\sigma_1 + \cdots + \sigma_n)$ is connected,

(2) every σ_j, $n+1 \leq j \leq m$, divides $F \setminus (\sigma_1 + \cdots + \sigma_n)$. We have

$$n - 1 \leq \varrho^+(F). \tag{9.31}$$

We remark that σ_j, $n+1 \leq j \leq m$, divides $F \setminus (\sigma_1 + \cdots + \sigma_n)$ into $m - n + 1$ parts $G_0, \ldots, G_{m-n}$. We may assume that the boundary G_0 contains σ_n. Among $G_1, \ldots, G_{m-n}$, there exists at least one part whose boundary contains only one cross-cut except $\sigma_1, \ldots, \sigma_n$. We denote this part G_1 and the cross-cut σ_{n+1}. Also among $G_2, \ldots, G_{m-n}$, there exists at least one part whose boundary contains only one cross-cut except $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}$. We denote this part G_2 and the cross-cut σ_{n+2}, and so on. Thus, we have $G_0, G_1, \ldots, G_{m-n}$ and $\sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_m$.

Now each G_k is a covering surface of B'. The boundary of G_k contains σ_{n+k}. Let S_k be the mean sheet number and let L_k be the length of the relative boundary. By Lemma 9.11, we have

$$\lambda(\sigma_{n+k}) \leq S_k + \frac{2^{18}q^3}{\varepsilon} L_k.$$

Hence, we have

$$\sum_{k=0}^{m-n} \lambda(\sigma_{n+k}) \leq S + \frac{2^{18}q^3}{\varepsilon} L. \tag{9.32}$$

On the other hand, by Lemma 9.12, we have

$$(q-1)S \leq \sum_{j=1}^{m} \lambda(\sigma_j) + \frac{2^{18}q^4}{\varepsilon} L. \tag{9.33}$$

Using (9.32), we obtain

$$(q-2)S \leq \sum_{j=1}^{n-1} \lambda(\sigma_j) + \frac{2^{18}(q^4+q^3)}{\varepsilon} L.$$

By (9.30), we obtain

$$(q-2)S \leq n - 1 + \frac{2^{18}(q^4+q^3)}{\varepsilon} L.$$

Thus, by (9.31), we obtain our result.

Case 2: All σ_j divide F. In this case, σ_j, $1 \leq j \leq m$, divide F into $m+1$ parts $G_0, \ldots, G_m$. Among them, there exists at least one part whose boundary contains only one cross-cut. We denote this part by G_1 and the cross-cut σ_1. Also among $G_2, \ldots, G_m$, there exists at least one part whose boundary contains only one cross-cut except σ_1. We denote this part G_2 and the cross-cut σ_2, and so on. Thus, we have $G_1, \ldots, G_m$ and $\sigma_1, \ldots, \sigma_m$.

By Lemma 9.11, we have

$$\lambda(\sigma_k) \leq S_k + \frac{2^{18}q^3}{\varepsilon} L_k.$$

Hence, we have

$$\begin{aligned} \sum_{k=1}^{m} \lambda(\sigma_k) &\leq \sum_{k=1}^{m} S_k + \frac{2^{18}q^3}{\varepsilon} \sum_{k=1}^{m} L_k \\ &\leq S + \frac{2^{18}q^3}{\varepsilon} L. \end{aligned} \tag{9.34}$$

By (9.33), we obtain our estimate. □

CHAPTER 10

Proof of Proposition 6.3

We shall derive Proposition 6.3 from Proposition 9.3 to conclude the proof of Theorem xx. Since Proposition 9.3 only treats the case when $(\overline{\mathbb{C}}, a_1(x), \ldots, a_q(x))$ is $\frac{1}{2^{20}}$-thick, we need to decompose the general case into $\frac{1}{2^{20}}$-thick cases. We use a similar trick as in [**18**] and [**17**] based on combinatorial arguments of trees.

10.1. Combinatorial lemma

A q-tail of a tree Γ is a map $\partial : \{1, \ldots, q\} \to \mathrm{vert}(\Gamma)$, where $\mathrm{vert}(\Gamma)$ is the set of vertices of Γ. For $v \in \mathrm{vert}(\Gamma)$, we set

$$\begin{aligned} P_v^m &= \{i \in \{1, \ldots, q\}\,; \partial(i) = v\}\,, \\ P_v^n &= \{v' \in \mathrm{vert}(\Gamma_x); v \text{ and } v' \text{ are adjacent }\}\,, \\ P_v &= P_v^m \cup P_v^n. \end{aligned} \tag{10.1}$$

We say that (Γ, ∂) is stable if $\#P_v \geq 3$ for all $v \in \mathrm{vert}(\Gamma)$.

Assume (Γ, ∂) is stable. For each $\tau \in P_v$, we define a subset $S_\tau^v \subset \{1, \ldots, q\}$ as follows. If $\tau \in P_v^m$, then we set $S_\tau^v = \{\tau\}$. For $\tau \in P_v^n$, we remove the edge $\{v, \tau\}$ from Γ to obtain two connected components Γ_v and Γ_τ, where Γ_v contains v and Γ_τ contains τ. We set

$$S_\tau^v = \{i \in \{1, \ldots, q\}\,; \partial(i) \in \mathrm{vert}(\Gamma_\tau)\}\,.$$

We define a map $\iota_v : P_v \to \{1, \ldots, q\}$ by

$$\iota_v(\tau) = \max S_\tau^v.$$

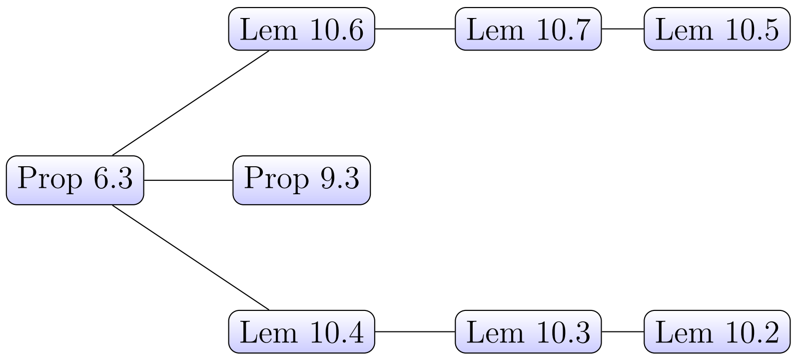


FIGURE 10.1. Proof of 6.3.

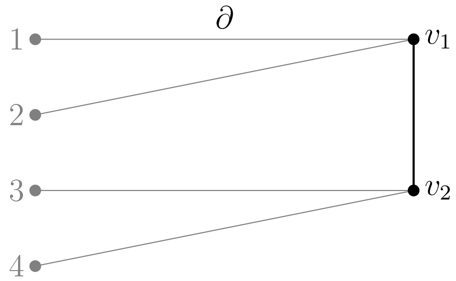


FIGURE 10.2. Example of a tree.

For each $v \in \mathrm{vert}(\Gamma)$, we have $q \in \iota_v(P_v)$. Let $j \in \iota_v(P_v) \setminus \{q\}$ be the largest element. We set

$$\iota_v(P_v)' = \iota_v(P_v) \setminus \{j, q\} .$$

EXAMPLE 10.1. (1) $\mathrm{vert}(\Gamma) = \{v\}$, $\partial : \{1,2,3\} \to \{v\}$. The set of edges of Γ is empty. In this case, we have $P_v^m = \{1,2,3\}$ and $P_v^n = \emptyset$, hence $P_v = \{1,2,3\}$. Thus, (Γ, ∂) is stable. We have

$$S_1^v = \{1\} \quad S_2^v = \{2\} \quad S_3^v = \{3\}$$

and

$$\iota_v(1) = 1 \quad \iota_v(2) = 2 \quad \iota_v(3) = 3.$$

Thus, $\iota_v(P_v)' = \{1\}$.

(2) $\mathrm{vert}(\Gamma) = \{v_1, v_2\}$ and $\partial : \{1,2,3,4\} \to \{v_1, v_2\}$, where

$$\partial(1) = v_1, \quad \partial(2) = v_1, \quad \partial(3) = v_2, \quad \partial(4) = v_4.$$

The set of edges of Γ consists of one edge which joins v_1 and v_2. In this case, we have

$$\begin{aligned} P_{v_1}^m &= \{1,2\} , \quad P_{v_1}^n = \{v_2\} , \quad P_{v_1} = \{1,2,v_2\} , \\ P_{v_2}^m &= \{3,4\} , \quad P_{v_2}^n = \{v_1\} , \quad P_{v_2} = \{3,4,v_1\} . \end{aligned}$$

Thus (Γ, ∂) is stable. We have

$$\begin{aligned} S_1^{v_1} &= \{1\} , \quad S_2^{v_1} = \{2\} , \quad S_{v_2}^{v_1} = \{3,4\} , \\ S_3^{v_2} &= \{3\} , \quad S_4^{v_2} = \{4\} , \quad S_{v_1}^{v_2} = \{1,2\} . \end{aligned}$$

Hence,

$$\begin{aligned} \iota_{v_1}(1) &= 1, \quad \iota_{v_1}(2) = 2, \quad \iota_{v_1}(v_2) = 4, \\ \iota_{v_2}(3) &= 3, \quad \iota_{v_2}(4) = 4, \quad \iota_{v_2}(v_1) = 2. \end{aligned}$$

Thus, $\iota_{v_1}(P_{v_1})' = \{1\}$ and $\iota_{v_2}(P_{v_2})' = \{2\}$.

LEMMA 10.2. *Assume (Γ, ∂) is stable. Then we have the disjoint union*

$$\{1, \ldots, q-2\} = \bigcup_{v \in vert(\Gamma)} \iota_v(P_v)'.$$

Proof. The inclusion

$$\bigcup_{v\in \mathrm{vert}(\Gamma)} \iota_v(P_v)' \subset \{1,\ldots,q-2\}$$

is obvious. We prove that for each $i \in \{1,\ldots,q_2\}$, there is a unique $v \in \mathrm{vert}(\Gamma)$ such that $i \in \iota_v(P_v)'$. Set $\partial(q) = v_o$ and $\partial(i) = v'$. Then there exists a unique path joining v_o and v':

$$v_o = v_0, v_1, \ldots, v_r = v'. \tag{10.2}$$

We set $k = \min\{s; i \in \iota_{v_s}(P_{v_s}\}$. We remark that

$$i \in \iota_{v_k}(P_{v_k})'. \tag{10.3}$$

This follows by $q-1 \in \iota_{v_o}(P_{v_o})$ if $k=0$. If $k \geq 1$, by $i \notin \iota_{v_{k-1}}(P_{v_{k-1}})$, we have $\iota_{v_{k-1}}(v_k) > i$ and $\iota_{v_{k-1}}(v_k) \in \iota_{v_k}(P_{v_k})$. Hence, we obtain (10.3).

Next we show the uniqueness. First, we take a vertex w outside the path (10.2). Then for $\tau \in P_w$ with $q \in S^w_\tau$, we have $i \in S^w_\tau$. Hence $i \notin \iota_w(P_w)$. Next, we consider the vertices in the path (10.2). Obviously, $i \notin \iota_{v_s}(P_{v_s})$ for $s < k$. For $s > k$, by $i \in \iota_{v_{s-1}}(P_{v_{s-1}})$, we have $i = \max \iota_{v_s}(P_{v_s} \setminus \{q\}$. Hence $i \notin \iota_{v_s}(P_{v_s})'$. This shows the uniqueness. □

10.2. Construction of a tree

Lemma 10.3. *For $x \in X(a_1,\ldots,a_q)$, there exists a stable q-tailed tree (Γ,∂) such that the following conditions hold:*

(1) For all $v \in$ vert(Γ), the marked sphere $(\overline{\mathbb{C}}, \left\{a_{\iota_v(\tau)}(x)\right\}_{\tau\in P_v}$ is $\frac{1}{2^{20}}$-thick,

(2) If v and v' are adjacent, then there exists an annulus A with modulus greater than $(1/2\pi)\log(2^{20})$ such that $(\overline{\mathbb{C}}, \left\{a_{\iota_v(\tau)}(x)\right\}_{\tau\in P_v\setminus\{v'\}}$ is contained in one component $\overline{\mathbb{C}} \setminus A$ and $(\overline{\mathbb{C}}, \left\{a_{\iota_{v'}(\tau)}(x)\right\}_{\tau\in P_{v'}\setminus\{v\}}$ is contained in the other component.

Proof. Starting from the q-tailed tree $(\Gamma^{[1]}, \partial^{[1]})$ defined by vert$(\Gamma^{[1]} = \{pt\}$, we consider the following algorithm:

1: If a q-tailed tree $(\Gamma^{[k]}, \partial^{[k]})$ satisfies the condition (1), then output $(\Gamma^{[k]}, \partial^{[k]})$. Otherwise go to the next step.

2: Find $v \in \mathrm{vert}(\Gamma^{[k]})$ such that the marked sphere

$$\left(\overline{\mathbb{C}}, \left\{a_{\iota^{[k]}_v(\tau)}(x)\right\}_{\tau\in P^{[k]}_v}\right)$$

is not $\frac{1}{2^{20}}$-thick. Thus there exists an annulus A with $\mathrm{Mod}(A) \geq (1/2\pi)\log 2^{20}$ that separates $P^{[k]}_v$ to $(P^{[k]}_v)'$ and $(P^{[k]}_v)''$. We construct a neq q-tailed tree $(\Gamma^{[k+1]}, \partial^{[k+1]})$ by replacing v with two new vertices v' and v'' such that $P^{[k+1]}_{v'} = (P^{[k]}_v)' \cup \{v''\}$ and $P^{[k+1]}_{v''} = (P^{[k]}_v)'' \cup \{v'\}$. Return to the previous step.

Note that each $(\Gamma^{[k]}, \partial^{[k]})$ is stable. Hence, the above procedure terminates at most in q steps and yields the desired stable q-tailed tree (Γ, ∂). □

We summarize the conclusion of Proposition 9.3 applied to $\{a_i\}_{i\in\iota_v}(P_v)$ as the set of rational functions. For each $v \in \mathrm{vert}(\Gamma)$, we apply Proposition 8.3 to obtain a holomorphic motion $\hat{\phi}_v : \Omega\times\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which extends $\{a_i\}_{i\in\iota_v}(P_v)$. Let $g_v : \Omega \to \overline{\mathbb{C}}$ be the quasimeromorphic function defined by

$$\hat{\phi}_v(\lambda, g_v) = f_v.$$

For $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, we define $\Delta^v_\tau \subset \overline{\mathbb{C}}$ as follows: Let $s < 1/10$.

If $\iota_v(\tau) \neq q$, then we set

$$\Delta^v_\tau = \left\{z \in \mathbb{C}; |z - a_{\iota_v(\tau)}(x)| < s\rho^v_\tau\right\},$$

where

$$\rho^v_\tau = \min_{i\in\iota_v(P_v)\setminus\{\iota_v(\tau),q\}} |a_i(x) - a_{\iota_v(\tau)}(x)|.$$

When $\iota_v(\tau) = q$, let j be the maximal element in $\iota_v(P_v) \setminus \{q\}$. We set

$$\Delta^v_\tau = \{z \in \mathbb{C}; |z - a_j(x)| > R_v/s\},$$

where

$$R_v = \max_{i\in\iota_v(P_v)\setminus\{q\}} |a_i(x) - a_j(x)|.$$

We set

$$\begin{aligned} B_v &= \overline{\mathbb{C}} \setminus \bigcup_{\tau\in P_v} \overline{\Delta^v_\tau}, \\ \chi_v(r,t) &= \sum_{F\in\mathcal{I}(g_v^{-1}(B_v),\Omega(r,t))} \varrho(F)+ = \sum_{F\in\mathcal{P}(g_v^{-1}(B_v),\Omega(r,t))} \varrho(F)^+. \end{aligned} \tag{10.4}$$

Then, by Proposition 9.3, we have

$$\begin{aligned} &(\#P_v - 2)\int_0^{m/2} T\left(r, \frac{f-a_i}{a_j-a_i}, \Omega(t)\right) dt \\ &\leq \int_0^{m/2}\int_1^r \frac{\chi_v(u,t)}{u}\, du dt + \frac{2^{67}dq^8}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \end{aligned} \tag{10.5}$$

for $r > \gamma_d$, where i and j are distinct elements in $\iota_v(P_v) \setminus \{q\}$.

So far we have assumed $s < 1/10$. In what follows, we take s so that $\frac{1}{16} < s < \frac{1}{10}$. Then we have the following lemma:

LEMMA 10.4. *For $v \in vert(\Gamma_x)$ and $\tau \in P_v$, we have $a_i(x) \in \Delta^v_\tau$ for all $i \in S^v_\tau$.*

PROOF. The assertion is obvious if $\tau \in P_v^m$. In the following, we assume that $\tau \in P_v^n$. It is enough to show

$$\overline{\mathbb{C}} \setminus \Delta_{v'}^{v} \cap \overline{\mathbb{C}} \setminus \Delta_{v}^{v'} = \emptyset \tag{10.6}$$

for adjacent vertices v and v'. Indeed (10.6) implies $\Delta_{v'}^{v} \supset \Delta_{\eta}^{v'}$ for all $\eta \in P_{v'} \setminus \{v\}$. We take the path joining v and $\partial(i)$:

$$v = v_0, v_1, \ldots, v_r = \partial(i).$$

Then we have

$$\Delta_{v_1}^{v_0} \supset \Delta_{v_2}^{v_1} \supset \cdots \supset \Delta_{v_r}^{v_{r-1}} \supset \Delta_{i}^{v_r} \ni a_i(x)$$

as desired.

We prove (10.6). We note that $S_{v'}^{v} \cup S_{v}^{v'} = \{1, 2, \ldots, q\}$ is a disjoint union. Hence, we may assume without loss of generality $q \in S_{v}^{v'}$. Thus, $\iota_{v'}(v) = q$. Set $\iota_v(v') = j$. We take $k \in \iota_v(P_v)$ such that $\rho_{v'}^{v} = |a_k(x) - a_j(x)|$. Then by $s > 1/16$, we have

$$\Delta_{v'}^{v} \supset \left\{z \in \overline{\mathbb{C}}; |\operatorname{cr}(z, a_j(x), a_k(x), a_q(x))| < 1/16\right\}. \tag{10.7}$$

We note that j is the largest element in $\iota_{v'}(P_{v'}) \setminus \{q\}$. We take $l \in \iota_{v'}(P_{v'})$ such that $R_{v'} = |a_l(x) - a_j(x)|$. Then by $s > 1/16$, we have

$$\Delta_{v}^{v'} \supset \left\{z \in \overline{\mathbb{C}}; |\operatorname{cr}(z, a_j(x), a_l(x), a_q(x))| > 16\right\}. \tag{10.8}$$

By Lemma 10.3, there exists an annulus A with modulus greater than $(1/2\pi)\log(2^{20})$ such that $\{a_k(x), a_q(x)\}$ is contained in one component of $\overline{\mathbb{C}} \setminus A$ and $\{a_j(x), a_l(x)\}$ is contained in the other component. Hence, by Teichmüller's extremal problem [2,p. 30], we have

$$|\operatorname{cr}(a_l(x), a_j(x), a_k(x), a_q(x)| < \frac{1}{2^{15}}. \tag{10.9}$$

By (10.7)–(10.9), we obtain (10.6). □

We apply the following distortion estimate of quasiconformal mappings [3,p. 81] to prove a generalization of (10.6): For a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 with $K_\psi < \frac{51}{49}$, we have

$$|\psi(z)| < 4|z|^{49/51}, \quad |z| < 1. \tag{10.10}$$

LEMMA 10.5. *For adjacent vertices v and v', and for all $\lambda \in \Omega$, we have*

$$\hat{\phi}_v(\lambda, \overline{\mathbb{C}} \setminus \Delta_{v'}^{v}) \cup \hat{\phi}_{v'}(\lambda, \overline{\mathbb{C}} \setminus \Delta_{v}^{v'}) = \emptyset. \tag{10.11}$$

PROOF. We keep the notation of the proof of (10.6). Let $\lambda \in \Omega$. We denote $\hat{\phi}_{v,\lambda}(z) = \hat{\phi}_v(\lambda, z)$. By Proposition 8.3, we have $K(\hat{\phi}_{v,\lambda}) < \frac{51}{49}$. We apply (10.10) to a quasiconformal map

$$\varphi_{a_j(x),a_k(x)} \circ \hat{\phi}_{v,\lambda}^{-1} \circ \varphi_{a_j(\lambda),a_k(\lambda)}^{-1},$$

where we recall the notation from (9.1). Note that this map fixes 0 and 1. Then we obtain

$$\hat{\phi}_{v,\lambda}^{-1}\left(\left\{z\in\overline{\mathbb{C}}; |\operatorname{cr}(z,a_j(\lambda),a_k(\lambda),a_q(\lambda)|\le 2^{-7}\right\}\right)$$
$$\subset\left\{z\in\overline{\mathbb{C}}; |\operatorname{cr}(z,a_j(x),a_k(x),a_q(x)|\le 2^{-4}\right\}.$$

Hence, we obtain

$$\hat{\phi}_{v,\lambda}(\overline{\mathbb{C}}\setminus\Delta_{v'}^{v})\subset\left\{z\in\overline{\mathbb{C}}; |\operatorname{cr}(z,a_j(\lambda),a_k(\lambda),a_q(\lambda)|> 2^{-7}\right\}. \tag{10.12}$$

Similarly, we have

$$\hat{\phi}_{v',\lambda}(\overline{\mathbb{C}}\setminus\Delta_{v}^{v'})\subset\left\{z\in\overline{\mathbb{C}}; |\operatorname{cr}(z,a_j(\lambda),a_k(\lambda),a_q(\lambda)|< 2^{7}\right\}. \tag{10.13}$$

By Lemma 7.3 and (10.9), we have

$$|\operatorname{cr}(a_l(\lambda),a_j(\lambda),a_k(\lambda),a_q(\lambda))|<\frac{1}{2^{14}}. \tag{10.14}$$

By (10.12)–(10.14), we establish our lemma. □

10.3. Final reduction

For $\tau\in P_v$, we set

$$a_\tau^v(r,t)=-\sum_{F\in\mathcal{D}_{v,\tau}^I}\varrho(F)-\sum_{F\in\mathcal{D}_{v,\tau}^P}\varrho^+(F),$$

where

$$\mathcal{D}_{v,\tau}^I=\mathcal{I}(g_v^{-1}(\Delta_\tau^v),\Omega(r,t)),\quad \mathcal{D}_{v,\tau}^P=\mathcal{P}(g_v^{-1}(\Delta_\tau^v),\Omega(r,t)).$$

It is evident that $\mathcal{D}_{v,\tau}^I$ is a finite set. We remark that $\mathcal{D}_{v,\tau}^P$ is also finite, since $\Omega(r,t)$ is bounded by a finite number of analytic arcs and g_v is real analytic outside the inverse image of $\{a_i(x)\}_{i\in\iota_v(P_v)}$.

By changing s slightly if necessary, we assume that: For all v and $\tau\in P_v$, if $g_v:\Omega\to\overline{\mathbb{C}}$ is non-constant, then g_v does not have branch points over $\partial\Delta_\tau^v$.

LEMMA 10.6.

$$\chi_v(r,t)\le\sum_{t\in\mathcal{P}_v}\alpha_\tau^v(r,t). \tag{10.15}$$

If v and v' are adjacent, then

$$\chi_v(r,t)\le-\alpha_v^{v'}(r,t)+\sum_{\tau\in\mathcal{P}_v\setminus\{v'\}}\alpha_\tau^v(r,t). \tag{10.16}$$

Lemma (10.6) *implies Proposition 6.3.* Set $v_o=\partial(q)$. By (10.15), we have

$$\chi_{vo}(r,t)\le\sum_{\tau\in\mathcal{P}_{v_o}}\alpha_\tau^{v_o}(r,t).$$

For $v \in \mathrm{vert}\Gamma_x \setminus \{v_o\}$, we denote by v^- the vertex with $q \in S^v_{v^-}$. By (10.16), we have

$$\chi_v(r,t) \le -\alpha_v^{v^-}(r,t) + \sum_{\tau \in \mathcal{P}_v \setminus \{v^-\}} \alpha^v_\tau(r,t).$$

Taking summation over all $v \in \mathrm{vert}(\Gamma_x)$, we obtain

$$\sum_{v\in \mathrm{vert}(\Gamma)} \chi_v(r,t) \le \sum_{v \in \mathrm{vert}(\Gamma_x)} \sum_{\tau \in P_v^m} \alpha^v_\tau(r,t). \tag{10.17}$$

Since $-\varrho(D) \le 1$ for $D \in \mathcal{D}^I_{v,\tau}$, we have

$$\begin{aligned}
&\sum_{v\in \mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \alpha^v_\tau(r,t) \\
&- \sum_{v\in \mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \sum_{D\in \mathcal{D}^I_{v,\tau}} \varrho(D) \\
&\le \sum_{v\in \mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \overline{n}(g_v, a_\tau(x), \Omega(r,t)).
\end{aligned}$$

Since $\overline{n}(g_v, a_\tau(x), \Omega(r,t)) = \overline{n}(f, a_\tau, \Omega(r,t))$, see (8.2), we obtain

$$\begin{aligned}
&\sum_{v\in \mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \alpha^v_\tau(r,t) \\
&\le \sum_{v\in \mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \overline{n}(g_v, a_\tau(x), \Omega(r,t)) \\
&= \sum_{i=1}^q \overline{n}(f, a_i, \Omega(r,t)).
\end{aligned}$$

Hence by (10.17), we have

$$\sum_{v\in \mathrm{vert}(\Gamma_x)} \chi_v(r,t) \le \sum_{i=1}^q \overline{n}(f, a_i, \Omega(r,t)). \tag{10.18}$$

Next we apply Proposition 9.3. For each $i \in \iota_v(P_v)'$, we take $\tau \in P_v$ such that $i = \iota_v(\tau)$. By $i < i^\diamondsuit$, we have $i^\diamondsuit \notin S^v_\tau$. Hence, we may take $\tau' \in P_v$ with $\tau' \ne \tau$ such that $i^\diamondsuit \in S^v_{\tau'}$. Let $j = \max \iota_v(P_v) \setminus \{q\}$. Then $j \ne i$. By

$$|a_i^\diamondsuit(x) - a_i(x)| \le |a_j(x) - a_i(x)|$$

and Lemma 10.4, we have $q \notin S^v_{\tau'}$. Set $i^\blacklozenge = \iota_v(\tau')$. Then $i < i^\blacklozenge < q$. Hence applying (10.5) to $i, i^\blacklozenge$ and taking average over $i \in \iota_v(P_v)'$, we obtain

$$\begin{aligned}
&\sum_{i\in\iota_v(P_v)'} \int_0^{m/2} T\left(r, \frac{f-a_i}{a_i^\blacklozenge - a_i}, \Omega(t)\right) dt \\
&\le \int_0^{m/2} \int_1^r \frac{\chi_v(u,t)}{u}\, du dt + \frac{2^{67} d q^8}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}
\end{aligned}$$

for $r > \gamma_d$. Thus, by Lemma 10.2 and (10.18), we have

$$\begin{aligned}&\sum_{i=1}^{q-2}\int_0^{m/2} T\left(r, \frac{f-a_i}{a_i^{\blacklozenge}-a_i}, \Omega(t)\right)\, dt\\ &\leq \sum_{i=1}^{q}\int_0^{m/2} \overline{N}(f, a_i, \Omega(t))\, dt\\ &+ \frac{2^{67dq^9}}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for $r > \gamma_d$.

Finally, we show the following estimate to conclude the proof.

$$\begin{aligned}&\int_0^{m/2} T\left(r, \frac{f-a_i}{a_i^{\lozenge}-a_i}, \Omega(t)\right)\, dt\\ &\leq \int_0^{m/2} T\left(r, \frac{f-a_i}{a_i^{\blacklozenge}-a_i}, \Omega(t)\right)\, dt\\ &+ 2^{26} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}.\end{aligned} \tag{10.19}$$

For the proof, we set

$$\kappa = \kappa(f, a_i, a_i^{\lozenge}, a_i^{\blacklozenge}).$$

Then by Lemma 7.2, we have

(10.20)

$$\begin{aligned}&\int_0^{m/2} \left| T\left(r, \frac{f-a_i}{a_i^{\lozenge}-a_i}, \Omega(t)\right) + TT\left(r, \frac{f-a_i^{\lozenge}}{a_i^{\blacklozenge}-a_i^{\lozenge}}, \Omega(t)\right) - T(r, \kappa, \Omega(r,t))\right|\, dt\\ &\leq 2^{25} T\left(r + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for $r > \gamma_d$. By the definition of $i^{\lozenge}$ and Lemma 10.4, we have

$$\frac{3}{4}|a_i^{\blacklozenge}(x) - a_i(x)| \leq |a_i^{\lozenge}(x) - a_i(x)|.$$

Hence, by Lemma 7.2, we have

(10.21)

$$\begin{aligned}&\int_0^{m/2} \left| T\left(r, \frac{f-a_i}{a_i^{\blacklozenge}-a_i}, \Omega(t)\right) + TT\left(r, \frac{f-a_i^{\lozenge}}{a_i^{\blacklozenge}-a_i^{\lozenge}}, \Omega(t)\right) - T(r, \kappa, \Omega(r,t))\right|\, dt\\ &\leq 2^{25} T\left(r + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for $r > \gamma_d$. This shows (10.19), and concludes the derivation of Proposition 6.3.

10.4. End of the proof

We prove Lemma 10.6 to finish the proof of Theorem 9.4 (Theorem 1.2). First we show (10.15). Let W be a connected component of $\Omega(r,t)$. We remark that

$$\varrho^+(W) = 0. \tag{10.22}$$

Indeed by $\infty \notin \Omega(t)$, each connected component of $\mathbb{C} \setminus W$ has nontrivial intersection with $\mathbb{C} \setminus \Omega(t)$. Hence by $\mathbb{C} \setminus \Omega(t) \subset \mathbb{C} \setminus W$, we conclude $\varrho(W) \leq \varrho(\Omega(t))$. This proves (10.22).

We need one lemma from [33,Lemma 1].

Lemma 10.7. *Assume that a finite number of points of disjoint simple closed curves γ_i $(i = 1, \ldots, p)$ divide $\overline{\mathbb{C}}$ into connected domains $D_1, \ldots, D_{p+1}$. Let $\zeta : W \to \overline{\mathbb{C}}$ be a covering map with no branch points over the boundaries of D_i $(1 \leq i \leq p+1)$. Put*

$$\mathcal{A} = \bigcup_{i=1}^{p+1} \mathcal{I}(\zeta^{-1}(D_i), W), \quad \mathcal{B} == \bigcup_{i=1}^{p+1} \mathcal{P}(\zeta^{-1}(D_i), W).$$

Then we have

$$\varrho^+(W) \geq \sum_{A \in \mathcal{A}} \varrho(A) + \sum_{B \in \mathcal{B}} \varrho^+(B).$$

Now to prove (10.15), we remark that the estimate is trivial if g_v is constant, since both sides are 0 by (10.22). When g_v is non-constant, by Lemma 10.7 and (10.22), we obtain (10.15).

Next we prove (10.16). We remark that

$$g_v(D) \not\subset \Delta^v_{v'} \quad \text{for} \quad D \in \mathcal{D}^I_{v',v}. \tag{10.23}$$

Indeed assume contrary that there exists $D \in \mathcal{D}^I_{v',v}$ such that $g_v(D) \subset \Delta^v_{v'}$. Then there exists $z \in D$ such that $g_{v'}(z) = a_{\iota_{v'}(v)}(x)$, which says $f(z) = a_{\iota_{v'}(v)}(z)$. Since $\iota_{v'}(v) \in \iota_v(P_v)$, we have $g_v(z) = a_{\iota_{v'}(v)}(x)$. Since $\iota_{v'}(v) \neq \iota_v(v')$, we have $a_{\iota_{v'}(v)}(x) \notin \Delta^v_{v'}$. This contradicts $g_v(D) \subset \Delta^v_{v'}$. Thus, (10.23) is proved.

We prove (10.16) in two cases.

Case 1: g_v is constant. In this case, it is enough to show $\mathcal{D}^I_{v',v} = \emptyset$, for we have $\chi_v(r,t) = 0$ and $\alpha^v_\tau(r,t) = 0$ by (10.22). Suppose, on contrary, there exists $D \in \mathcal{D}^I_{v',v}$. Then by (10.23), $g_v(D) \not\subset \Delta^v_{v'}$. Since g_v is constant, we obtain

$$g_v(\Omega) \subset \overline{\mathbb{C}} \setminus \Delta^v_{v'}.$$

On the other hand, by (10.11), we have

$$g_{v'}(g_v^{-1}(\overline{\mathbb{C}} \setminus \Delta^v_{v'})) \subset \Delta^{v'}_v.$$

Hence, we conclude $g_{v'}(\Omega) \subset \Delta^{v'}_v$, which implies $\mathcal{D}^I_{v',v} = \emptyset$. This is a contradiction. Thus, we have proved (10.16) when g_v is constant.

Case 2: g_v is non-constant. Given $H \in \mathcal{D}^I_{v',v} \cup \mathcal{D}^P_{v',v}$, we consider the restriction $g_v|_H : H \to \overline{\mathbb{C}}$. We set

$$\begin{aligned} \mathcal{D}^I_{v,\tau,H} &= \mathcal{I}(g_v^{-1}(\Delta^v_\tau), H), \quad \mathcal{D}_{v,\tau,H} = \mathcal{P}(g_v^{-1}(\Delta^v_\tau), H), \quad \tau \in P_v \\ \mathcal{F}^I_{v,H} &= \mathcal{I}(g_v^{-1}(B_v), H), \quad \mathcal{F}_{v,H} = \mathcal{P}(g_v^{-1}(B_v), H). \end{aligned}$$

We first remark that

$$\mathcal{D}^I_{v,v',H} = \emptyset. \tag{10.24}$$

To show this, we assume on contrary that there exists $D \in \mathcal{D}^I_{v,v',H}$. By the same reason with (10.23), we have $g'_v(D) \neq\subset \Delta^{v'}_v$. On the other hand, we have $g_{v'}(H) \subset \Delta^{v'}_v$, for $H \in \Delta^I_{v',v} \cup \Delta^P_{v',v}$. This is a contradiction. Thus we have proved (10.24).

Now let us fix a component $H \in \mathcal{D}^I_{v',v}$. We have

$$\begin{aligned} \varrho(H) \geq \sum_{\tau \in P_v \setminus \{v'\}} \sum_{D \in \mathcal{D}^I_{v,\tau,H} \cup \mathcal{D}^P_{v,\tau,H}} \varrho(D) + \sum_{D \in \mathcal{D}^I_{v,v',H} \cup \mathcal{D}^P_{v,v',H}} \varrho(D) \\ + \sum_{F \in \mathcal{F}^I_{v,H} \cup \mathcal{F}^P_{v,H}} \varrho(F). \end{aligned}$$

Since H is compactly contained in $\Omega(r,t)$, the boundary ∂H of H does not meet the boundary of $\Omega(r,t)$. By (10.11), we have

$$g_v(g_{v'}^{-1}(\overline{\mathbb{C}} \setminus \Delta^{v'}_v)) \subset \Delta^v_{v'}. \tag{10.25}$$

Hence, we have

$$g_v(\partial H) \subset \Delta^v_{v'}.$$

Hence, $\mathcal{F}^P_{v,H} = \emptyset$ and $\mathcal{D}^P_{v,\tau,H} = \emptyset$ for $\tau \in P_v \setminus \{v'\}$. By (10.23), $g_v(H) \not\subset \Delta^v_{v'}$. Hence components D in $\mathcal{D}^P_{v,v',H}$ are not simply connected, so $\varrho(D) \geq 0$. Thus by (10.24) we obtain

$$\varrho(H) = \sum_{\tau \in P_v \setminus \{v'\}} \sum_{D \in \mathcal{D}^I_{v,t,H}} \varrho(D) + \sum_{F \in \mathcal{F}^I_{v,H}} \varrho(F). \tag{10.26}$$

Next we fix a component $H \in \mathcal{D}^P_{v',v}$. By Lemma 10.7 and (10.24), we have

$$\begin{aligned} \varrho^+(H) \geq \sum_{\tau \in P_v \setminus \{v'\}} \left(\sum_{D \in \mathcal{D}^I_{v,\tau,H}} \varrho(D) + \sum_{D \in \mathcal{D}^P_{v,\tau,H}} \varrho^+(D) \right) \\ + \sum_{F \in \mathcal{F}^I_{v,H}} \varrho(F) + \sum_{F \in \mathcal{F}^P_{v,H}} \varrho^+(F). \end{aligned} \tag{10.27}$$

Thus, by (10.26) and (10.27), we obtain

(10.28)

$$\sum_{H\in\mathcal{D}^I_{v',v}} \varrho(H) + \sum_{H\in\mathcal{D}^P_{v',v}} \varrho^+(H)$$

$$\geq \sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}} \sum_{\tau\in P_v\setminus\{v'\}} \sum_{D\in\mathcal{D}^I_{v,\tau,H}} \varrho(D) + \sum_{H\in\cup\mathcal{D}^P_{v',v}} \sum_{\tau\in P_v\setminus\{v'\}} \sum_{D\in\mathcal{D}^P_{v,\tau,H}} \varrho^+(D)$$

$$+ \sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}} \sum_{F\in\mathcal{F}^I_{v,H}} \varrho(F) + \sum_{H\in\cup\mathcal{D}^P_{v',v}} \sum_{F\in\mathcal{F}^P_{v,H}} \varrho^+(F).$$

By (10.25), we have

$$g_v^{-1}(B_v)\cap\Omega(r,t) \subset \bigcup_{H\in\mathcal{D}^I_{v',v}} H \cup \bigcup_{H\in\mathcal{D}_{v',v}} H.$$

Hence, we have

$$\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}} \sum_{F\in\mathcal{F}^I_{v,H}} \varrho(F) + \sum_{H\in\mathcal{D}^P_{v',v}} \sum_{F\in\mathcal{F}^P_{v,H}} \varrho^+(F) = \chi_v(r,t). \tag{10.29}$$

Again, by (10.25), we have

$$g_v^{-1}(\overline{\Delta^v_\tau})\cap\Omega(r,t) \subset \bigcup_{H\in\mathcal{D}^I_{v',v}} H\cup \subset \bigcup_{H\in\mathcal{D}^P_{v',v}} H, \text{ for } \tau\in P_v\setminus\{v'\}\,.$$

Hence, we have

(10.30)

$$\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}} \sum_{\tau\in P_v\setminus\{v'\}} \sum_{D\in\mathcal{D}^I_{v,\tau,H}} \varrho(D) + \sum_{H\in\mathcal{D}^P_{v',v}} \sum_{\tau\in P_v\setminus\{v'\}} \sum_{D\in\mathcal{D}^P_{v,\tau,H}} \varrho^+(D)$$

$$= -\sum_{\tau\in P_v\setminus\{v'\}} \alpha^v_\tau(r,t).$$

Thus, by (10.28)–(10.30), we obtain (10.16). $\square$

APPENDIX A

Appendix

A.1. Pullback

Let $f : A \to B$ and $g : B \to C$. In this case, the pullback of g by f is

$$f^*g = g \circ f.$$

The domain of definition of g was “pulled back” to A.

Consider

$$\int_B g(z)\, dm(z),$$

where $dm(z) = dxdy$ is the Lebesgue measure. Now

$$\int_B g(z)\, dm(z) = \int_A g(f(z))|f'(z)|^2\, dm(z) = \int_A f^*(g(z)\, dm(z))$$

Hence in this case

$$f^*(g\, dm(z)) = g(f(z))|f'(z)|^2\, dm(z).$$

If we set

$$\omega = g(z)\, dm(z),$$

we obtain

$$f^*\omega = g(f(z))|f'(z)|^2\, dm(z)$$

A.2. Derivatives

We have

$$\begin{aligned}
\frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log|f((re^{i\theta}))|d\theta &= \frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \operatorname{Re}\log(f((re^{i\theta})))d\theta \\
&= \frac{r}{2\pi}\frac{d}{dr}\operatorname{Re}\int_0^{2\pi} \log(f((re^{i\theta})))d\theta \\
&= \operatorname{Re}\frac{1}{2\pi i}\int_0^{2\pi} \frac{f'((re^{i\theta}))(re^{i\theta})id\theta}{f((re^{i\theta}))} \\
&= \operatorname{Re}\frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(z)}{f(z)}dz \\
&= \operatorname{Re}(n(r,0,f) - n(r,\infty,f)) \\
&= n(r,0,f) - n(r,\infty,f)
\end{aligned}$$

by the argument principle.

First, we want to show the identity

(A.1)
$$\frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi}\log|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi}\frac{f'(z)}{f(z)}dz = n(r,f,0)-n(r,f,\infty)$$

We see that

$$\begin{aligned}\frac{d}{dr}\log|f(re^{i\theta})| &= \frac{d}{dr}\frac{1}{2}\left(\log f(re^{i\theta}) + \log\overline{f(re^{i\theta})}\right)\\ &= \frac{1}{2}\left(\frac{f'(re^{i\theta})e^{i\theta}}{f(re^{i\theta})} + \frac{\overline{f'(re^{i\theta})e^{i\theta}}}{f(re^{i\theta})}\right)\end{aligned}$$

We obtain

$$\begin{aligned}\frac{r}{2\pi}\frac{d}{dr}\log|f(re^{i\theta})|d\theta &= \frac{r}{4\pi i}\left(\frac{f'(re^{i\theta})re^{i\theta}id\theta}{f(re^{i\theta})} - \frac{\overline{f'(re^{i\theta})re^{i\theta}id\theta}}{f(re^{i\theta})}\right)\\ &= \frac{r}{4\pi i}\left(\frac{f'(re^{i\theta})}{f(re^{i\theta})}dz - \frac{\overline{f'(re^{i\theta})}}{f(re^{i\theta})}d\overline{z}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi}\log|f(re^{i\theta}|d\theta &= \frac{r}{2\pi}\int_0^{2\pi}\frac{d}{dr}\log|f(re^{i\theta}|d\theta\\ &= \frac{1}{2}\left(\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\int_{|z|=r}\frac{\overline{f'(z)}}{\overline{f(z)}}d\overline{z}\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{f'(z)}{f(z)}dz\\ &= n(r,f,0)-n(r,f,\infty),\end{aligned}$$

where, in the end, we used the Argument principle. Hence (A.1) holds.

We have the notation

$$N(r,f,0) = \int_1^r \frac{n(t,f,0)}{t}dt.$$

We divide both sides of (A.1) with r and integrate from 1 to r to obtain

$$\frac{1}{2\pi}\int_1^r\int_0^{2\pi}\log|f(se^{i\theta})|d\theta ds = N(r,f,0)-N(r,f,\infty).$$

Example. Let g be an analytic function and $z = re^{i\theta}$. Now

(A.2)
$$\begin{aligned}2\frac{d}{dr}\log|g(re^{i\theta})| &= \frac{d}{dr}\left(\log g(re^{i\theta}) + \log\overline{g(re^{i\theta})}\right)\\ &= \frac{g'(re^{i\theta})e^{i\theta}}{g(re^{i\theta})} + \frac{\overline{g'(re^{i\theta})e^{i\theta}}}{g(re^{i\theta})}\\ &= 2\mathrm{Re}\frac{g'(re^{i\theta})e^{i\theta}}{g(re^{i\theta})}.\end{aligned}$$

We obtain

$$2\frac{d}{dr}\log|g(re^{i\theta})| = \frac{1}{|z|}2\mathrm{Re}\frac{g'(z)z}{g(z)}.$$

When g is meromorphic, $h(z) = g'(z)z/g(z)$ is analytic and $h(0) \neq 0, \infty$, since the possible pole or zero of g cancels out. The function

$$\mathrm{Re}\frac{g'(z)z}{g(z)}$$

is harmonic as the real part of an analytic function.

A.3. Modulus

A.4. Modulus of rectangle

Let $Q = [0, a] \times [0, b]$. Then the ratio

$$m(Q) = \frac{b}{a},$$

modulus of Q measures "how thick Q is". It is a good measure for thickness, since it is scale invariant: if $Q' = [0, 1] \times [0, b/a]$, then $m(Q') = m(Q)$, for example.

The conformal map φ

$$\varphi(z) = \exp\left(ze^{-i\frac{\pi}{2}}\frac{2\pi}{a}\right)$$

rotates Q clockwise 90 degrees, scales the height of the rectangle to be 2π and then maps the rectangle to an annulus

$$\varphi(Q) = A(0, 1, e^{2\pi\frac{b}{a}}),$$

where

$$A(z_0, r_1, r_2) = \{z \in \mathbb{C}\,;\, r_1 \leq |z - z_0| \leq r_2\}.$$

If we define the modulus of an annulus $A(z_0, r_1, r_2)$ to be

$$m(A(z_0, r_1, r_2)) = \frac{1}{2\pi}\log\frac{r_2}{r_1},$$

then

$$m(\varphi(Q)) = \frac{1}{2\pi}\log\frac{e^{2\pi\frac{b}{a}}}{1} = \frac{b}{a} = m(Q),$$

and φ preserves the modulus.

To define modulus for a general simply connected domain in the complex plane, we follow the idea that

$$m(Q) = \frac{b}{a} = \frac{ab}{a^2} = \frac{\mathrm{area}(Q)}{(\mathrm{length}(Q))^2}.$$

We will obtain so-called "length-area methods".

[From Fletcher, Markovic - Quasiconformal maps and Teichmüller theory]

Let $\Omega \subset \mathbb{C}$ be a domain. A metric $\rho(z)$ is admissible if ρ is measurable, $\rho(z) \geq 0$ and $A(\rho) \neq 0, \infty$, where

$$A(\rho) = \int_\Omega \rho^2(z)\, dm(z).$$

Let Γ be a family of rectifiable curves in Ω. For $\gamma \in \Gamma$, the length of γ with respect to ρ is

$$L_\gamma(\rho) = \int_\gamma \rho(z)|dz|$$

and we define

$$L(\rho) = \inf_{\gamma\in\Gamma} L_\gamma(\rho).$$

DEFINITION A.1. The extremal length of Γ is

$$\lambda(\Gamma) = \sup_\rho \frac{L^2(\rho)}{A(\rho)},$$

where the supremum is taken over admissible metrics ρ.

Extremal length can be thought of as an average minimal length[1] for curve families. We say that $\Gamma_1 < \Gamma_2$ if for all $\gamma_2 \in \Gamma_2$, there exists $\gamma_1 \in \Gamma_1$ such that $\gamma_1 \subset \gamma_2$.

LEMMA A.2. *If* $\Gamma_1 \leq \Gamma_2$, *then* $\lambda(\Gamma_1) < \lambda(\Gamma_2)$.

PROOF. Fix ρ, then $L_{\gamma_1} \leq L_{\gamma_2}$, so the inequality holds if the infimum is taken on both sides. Thus $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$. ☐

EXAMPLE A.3. Let Γ be the family of curves connecting vertical sides of a rectangle Q and not leaving Q. Fix ρ. Then

$$L(\rho) \leq \int_0^a \rho(x+iy)\, dx,$$

where the integral is over a horizontal curve.

$$(bL(\rho))^2 \leq \left(\int_Q \rho(z)\, dxdy\right)^2 \leq ab \int_Q \rho^2(z)\, dxdy$$

by using the Cauchy-Schwarz inequality. This then yields

$$\frac{L^2(\rho)}{A(\rho)} \leq \frac{a}{b}$$

[1]Let Γ be the family of curves connecting vertical sides of a rectangle Q. We want to show that

$$\lambda(\Gamma) = \frac{a^2}{ab} = \frac{a}{b}.$$

Then we will have

$$m(Q) = \frac{1}{\lambda(\Gamma)} = \frac{b}{a}.$$

So, $\lambda(Q)$ measures the "length of Q" and $m(Q) = 1/\lambda(Q)$ measures the "thickness of Q".

and so

$$\lambda(\Gamma) \leq \frac{a}{b}.$$

Take $\rho_0(z) = 1$ for $z \in Q$ and $\rho_0(z) = 0$ for $z \notin Q$, then

$$\frac{L^2(\rho_0)}{A(\rho_0)} = \frac{a}{b}.$$

Thus $\lambda(\Gamma) = a/b$.

Example A.4. Let $A = A(0, r_1, r_2) = \{z; r_1 \leq |z| \leq r_2\}$, and Γ be the family of curves that separate the two boundary circles, with the curves not leaving A.

Any path is longer than the circular path following the inner boundary component of A.

$$L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho) \leq \int_0^{2\pi} \rho((e^{i\theta}))d\theta$$

for some $r \in [r_1, r_2]$. Then

$$\left(\log\left(\frac{r_2}{r_1}\right) L(\rho)\right)^2 \leq \left(\int_{r_1}^{r_2} \int_0^{2\pi} \rho((re^{i\theta}))d\theta \frac{dr}{r}\right)^2 \leq \int_A \rho^2 \int_A d\theta dr$$

by the Cauchy-Schwarz inequality, and so

$$\frac{L^2(\rho)}{A(\rho)} \leq \frac{2\pi}{\log(r_2/r_1)}.$$

Take $\rho_0(z) = 1$ for $z \in A$ and 0 for $z \notin A$, then

$$\lambda(\Gamma) = \frac{L^2(\rho_0)}{A(\rho_0)} = \frac{2\pi}{\log(r_2/r_1)}.$$

A.4.1. Quasiconformal diffeomorphisms. Let $f : \Omega \to f(\Omega)$ be a C^1-diffeomorphism, that is, a bijection with continuous partial derivatives. Since f is a diffeomorphism, it is well approximated by a linear map. The linear map df maps the unit circle to an ellipse with major axis α and minor axis β. The area of $\mathbb{D}$ is π, and the area of $df(\mathbb{D}) = \pi\alpha\beta$, so the Jacobian $J_f(z_0) = |f_z|^2 - |f_{\overline{z}}|^2 = \alpha\beta$. Since sense-preserving maps are being considered here, $|f_z| > |f_{\overline{z}}|$ and so $J_f > 0$. The dilatation (or distortion) at z_0 is defined to be

$$D_f = \frac{\alpha}{\beta} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1.$$

The complex dilatation at z_0 is

$$\mu_f = \frac{f_{\overline{z}}}{f_z}.$$

The dilatation and distortion are related by

$$D_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}.$$

By the Cauchy-Riemann equations, if $f \in C^1$, then f is conformal if and only if $f_{\bar{z}} \equiv 0$. If f is conformal, $D_f = 1$ and $\alpha = \beta$ above, so df maps circles to circles.

DEFINITION A.5. Let $f : \Omega \to \mathbb{C}$ be a diffeomorphism. We say that f is a quasiconformal map if $D_f(z)$ is bounded in Ω. We say f is a K-quasiconformal map if

$$D_f(z) \leq K, \quad z \in \Omega. \tag{A.3}$$

K_f is defined to be the infimum of all such K.

If f satisfies (A.3), but is not bijective, then f is called K-*quasiregular*.

A C^1-diffeomorphism is conformal if and only if it is 1-quasiconformal, and so, in some sense, the larger K_f is, the further f is from a conformal map.

A.4.2. Modulus is a conformal invariant.

THEOREM A.6. *Let $f : \Omega \to f(\Omega)$ be a C^1 K-quasiconformal map, and let Γ be a family of curves which do not leave Ω. Let $\Gamma_1 = f(\Gamma) \subset f(\Omega)$. Then*

$$\frac{\lambda(\Gamma)}{K} \leq \lambda(\Gamma') \leq K\lambda(\Gamma).$$

PROOF. Let ρ be admissible for Γ and define

$$\rho_1(\zeta) = \left(\frac{\rho}{|f_z| - |f_{\bar{z}}|} \circ f^{-1}\right)(\zeta)$$

where $f(z) = \zeta$ and ρ_1 is admissible for Γ_1. Then

$$\int_{f(\Omega)} \rho_1^2 |d\zeta|^2 = \int_{\Omega} \rho^2 \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K_f A(\rho).$$

If $\gamma_1 = f(\gamma)$, for $\gamma \in \Gamma$ and $\gamma_1 \in \Gamma_1$, then

$$\int_{\gamma_1} \rho_1 |d\zeta| \geq \int_{\gamma} \rho |dz|$$

which implies

$$L_{\gamma_1}(\rho_1) \geq L_{\gamma}(\rho)$$

and hence

$$L(\rho_1) \geq L(\rho)$$

and so, $\lambda(\Gamma) \geq K_f^{-1}\lambda(\Gamma)$. Apply the same argument to f^{-1} for the other inequality. $\square$

A.5. Teichmüller spaces

A.5.1. Pants. A pair of pants is a set homeomorphic to

$$D(0,4) \setminus (D(2,1) \cup D(-2,1)).$$

A closed surface with g holes, that is, of genus g can be divided to

$$2g-2 \tag{A.4}$$

pants. To see this, note first that a double torus with 2 holes can be divided into 2 pants. Hence, in this case

$$2g-2=2$$

as the formula shows. On the other hand, if a surface has g holes, then one hole can be introduced by adding 2 pants. Hence, formula A.4 holds.

Each pant has 3 boundary components and on a surface, they are shared by 2 pants, so there are

$$(2g-2)\cdot\frac{3}{2} = 3g-3$$

cuts. Each cut is described by its length and an angle, which explains in which position the parts are joined. Therefore each cut is described by one complex parameter. We see that a surface of genus g depends on

$$3g-3$$

complex parameters.

The reasoning can be made exact by so-called Fenchel-Nielsen parametrization.

However, for this to be meaningful, we need to show that the dependance is somehow continuous.

A.5.2. Miscellanious. Uniformization theorem states that a simply connected Riemann surface is isomorphic to either the Riemann sphere $\overline{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$.

Here $\overline{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic to each other. Namely, $\overline{\mathbb{C}}$ is compact but $\mathbb{C}$ and $\mathbb{D}$ are not. Moreover, $f:\mathbb{D}\to\mathbb{C}$ can be non-constant and analytic, but $g:\mathbb{C}\to\mathbb{C}$ cannot, by Liouville's theorem.

A.5.3. Thin and thick. Hubbard writes [**7**, p. 86]

> "Bill Thurston and Dennis Sullivan have taught me that, roughly, one can think of a Riemann surface as made of standard plumbing joints connected by pipes that are cut to order and can be arbitrarily long. More precisely, every Riemann surface consists of a "thick part" with bounded geometry, corresponding to standard plumbing joints, and a "think part" that may be unbounded but is essentially simple – just a pipe. In

> this section we discuss a theorem of hyperbolic geometry, Theorem 3.8.3, which justifies this view."

A.6. Improvements to Yamanoi's estimates on $\widetilde{\alpha}$ and $\widetilde{\beta}$

Yamanoi did rough estimates

$$|\sin\theta| \le 1, \quad \log\frac{1}{\log\sigma} \le \frac{1}{\log\sigma}$$

and did not optimize the powers in $\delta = (\log\sigma)^4$ and $\tau = (\log\sigma)^{10}$ and $\lambda(r)^2$. We will optimize these things here.

Yamanoi got the estimate

$$\int_r^{\sigma r} v(t,f,\tau)\frac{dt}{t} \le \frac{1+\log\sigma}{\log\sigma}(2T(\sigma\rho,f)+c)(\widetilde{\alpha}(\tau)+\widetilde{\beta}(\tau)).$$

We can estimate Yamanoi's auxiliary functions better by

$$\widetilde{\alpha}(\tau) \le \frac{9\tau}{\log\sigma} \tag{A.5}$$

and

$$\widetilde{\beta}(\tau) \le 15\delta + 8\delta\log\frac{1}{\delta} + \frac{6\tau\log\sigma}{\delta}. \tag{A.6}$$

Also here, these estimates have been simplified for clarity and are far from sharp. The difference to Yamanoi's estimates is that we got rid of the powers 2 in the denominator in the original estimates

$$\widetilde{\alpha}(\tau) \le \frac{C\tau}{(\log\sigma)^2}, \quad \frac{10\tau\log\sigma}{\delta^2}.$$

To treat $\log\frac{1}{\delta}$ carefully, let $0<\eta<1$ and set

$$h_\eta(\sigma) = (\log\sigma)^\eta \log\frac{1}{\log\sigma}.$$

Now $h_\eta(\sigma)\to 0$ as $\sigma\to 1$.

Let $\delta = (\log\sigma)^p$ for $0<p\le 4$. Then by $\log\frac{1}{\delta} \le 4\frac{1}{\log\sigma}$, we have

(A.7)
$$\begin{aligned}
&\frac{1+\log\sigma}{\log\sigma}(\widetilde{\alpha}(\tau)+\widetilde{\beta}(\tau))\\
&\le \frac{18\tau}{(\log\sigma)^2} + 30(\log\sigma)^{p-1} + 64(\log\sigma)^{p-1}\log\frac{1}{\log\sigma} + 12\tau(\log\sigma)^{-p}\\
&\le \frac{18\tau}{(\log\sigma)^2} + 30(\log\sigma)^{p-1} + 64(\log\sigma)^{p-1-\eta}h_\eta(\sigma) + 12\tau(\log\sigma)^{-p}
\end{aligned}$$

We choose τ so that we get some power $(\log\sigma)^P$, $P>1$, to the right hand side. Hence, for $S>0$, we choose $\tau = (\log\sigma)^{3+S}$ and $p = 2+2S/3$ and $\eta = S/3$ to obtain

$$2\frac{1+\log\sigma}{\log\sigma}(\widetilde{\alpha}(\tau)+\widetilde{\beta}(\tau)) \le (120+128h(\sigma))(\log\sigma)^{1+S/3},$$

where $h(\sigma) \to 0$ as $\sigma \to 1$.

A.6.1. We can replace $(\log \sigma)^{10}$ **by** $(\log \sigma)^{3+S}$ **and** $\lambda(r)^{20}$ **by** $\lambda(r)^{3+S}$**, for** $0 < S < 1$**.** The rest of the proofs go in same fashion and we obtain

$$\int_r^{\sigma r} \frac{v(t, f, (\log \sigma)^{3+S})}{t} dt < 248(\log \sigma)^{1+S/3}(T(\sigma^3 r, f) + c)$$

for $\sigma > 1$ small enough and

$$v(r, f, \lambda(r)^{3+S}) \leq \varepsilon T(r, f)$$

for $S > 0$ by choosing $\sigma = \exp(\lambda(r)^s)$ for $s > 0$ small enough in the proof.

Proof of (A.5) *and* (A.6). Yamanoi's auxiliary function $\widetilde{\alpha}$ satisfies

$$\widetilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t, \tau) \frac{dt}{t} \leq 4 \int_{1/\sigma^2}^{1/\sigma} \int_0^{\tau/2} \frac{(1+t)(1-t)\sin\theta}{|1 - te^{i\theta}|^4} d\theta dt. \tag{A.8}$$

We estimate

$$|1 - te^{i\theta}| = |1 - t\cos\theta + it\sin\theta| = \sqrt{(1 - t\cos\theta)^2 + (t\sin\theta)^2} \geq t|\sin\theta|$$

and

$$|1 - te^{i\theta}|^3 \geq (1 - t)^3$$

and obtain

$$\begin{aligned} \widetilde{\alpha}(\tau) &\leq 2\tau \int_{1/\sigma^2}^{1/\sigma} \frac{1+t}{t(1-t)^2} dt \\ &= 2\tau \int_{1/\sigma^2}^{1/\sigma} \frac{1}{t} + \frac{1}{1-t} + \frac{2}{(1-t)^2} dt \\ &= 2\tau \, []_{t=1/\sigma^2}^{1/\sigma} \\ &= 2\tau \left(\log(\sigma + 1) + \frac{2\sigma}{(\sigma+1)(\sigma-1)} \right) \\ &\leq 3\tau + \frac{2\sigma\tau}{\sigma - 1} \leq 3\tau + \frac{2\sigma\tau}{\log\sigma} \leq \frac{9\tau}{\log\sigma} \end{aligned} \tag{A.9}$$

by

$$\log\sigma \leq \sigma - 1 \leq 2\log\sigma, \quad 1 < \sigma < e.$$

On the other hand, since $\beta(t, a, \tau) = \beta(t, |a|, \tau)$, we can assume that $0 < a < 1$. We have

$$\widetilde{\beta}(\tau) = \sup_{0<a<1} \int_{1/\sigma^2}^{1/\sigma} \beta(t, a, \tau) \frac{dt}{t}.$$

Let $\delta > 0$. As in Yamanoi's original reasoning, we set

$$\left[1/\sigma^2, 1/\sigma\right] = I(a) \cup J(a),$$

where

$$I(a) = \left[1/\sigma^2, 1/\sigma\right] \quad \cap \quad [|a| - \delta, |a| + \delta]$$
$$J(a) = \left[1/\sigma^2, 1/\sigma\right] \quad \setminus \quad [|a| - \delta, |a| + \delta],$$

and obtain

$$\int_{1/\sigma^2}^{1/\sigma} \beta(t, a, \tau)\frac{dt}{t} \leq \sigma^2 \int_{I(a)} \beta(t, a, \tau)dt + \int_{J(a)} \beta(t, a, \tau)\frac{dt}{t}. \tag{A.10}$$

Yamanoi's reasoning shows that

$$\int_{I(a)} \beta(t, a, \tau)dt \leq 2\delta + 2\delta \log \frac{1}{\delta}.$$

To estimate the second integral, we estimate the Green function. We have

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = at \sin\theta \left(\frac{1}{|1 - ate^{i\theta}|^2} - \frac{1}{|a - te^{i\theta}|^2}\right).$$

Here

$$|1 - ate^{i\theta}|^2 = (1 - at\cos\theta)^2 + (at\sin\theta)^2 = 1 - 2at\cos\theta + a^2t^2$$

and similarly

$$|a - te^{i\theta}|^2 = a^2 - 2at\cos\theta + t^2.$$

We obtain

$$|a - te^{i\theta}|^2 - |1 - ate^{i\theta}|^2 = (1 - a^2)(1 - t^2)$$

and hence

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = -\frac{(1 - a^2)(1 - t^2)at\sin\theta}{|1 - ate^{i\theta}|^2 |a - te^{i\theta}|^2}.$$

We estimate

$$|1 - ate^{i\theta}|^2 \geq (1 - at)(1 - at) \geq (1 - a)(1 - t)$$

and

$$|a - te^{i\theta}| = \sqrt{(a - t\cos\theta)^2 + (t\sin\theta)^2} \geq t|\sin\theta|.$$

Hence

$$\left|\frac{\partial g}{\partial \theta}(te^{i\theta}, a)\right| \leq \frac{(1 + a)a(1 + t)}{|a - te^{i\theta}|} \leq 2\frac{1 + t}{|a - t|}.$$

Hence, on $t \in J(a)$, we obtain

$$\beta(t, a, \tau) \leq \frac{2\tau}{\delta}(1 + t).$$

Thus, we obtain

$$\begin{aligned} \int_{J(a)} \beta(t, a, \tau)\frac{dt}{t} &\leq \frac{2\tau}{\delta} \int_{1/\sigma^2}^{1/\sigma} \left(\frac{1}{t} + 1\right) dt \\ &= \frac{2\tau}{\delta}\left(\frac{\sigma - 1}{\sigma} + \log\sigma\right) \leq \frac{6\tau\log\sigma}{\delta}, \end{aligned} \tag{A.11}$$

by

$$\log\sigma \le \sigma - 1 \le 2\log\sigma, \quad 1 < \sigma < e.$$

A.7. Hyperbolic geometry

A.7.1. Relation of hyperbolic and Euclidean metrics.

THEOREM A.7. *Let AB and CD be two chords of a circle intersecting at O. Then*

$$AO \cdot OB = CO \cdot OD.$$

PROOF. By the inscribed angle theorem,

$$\triangle AOD \sim \triangle COB$$

and thus

$$\frac{AO}{OD} = \frac{CO}{OB}.$$

□

THEOREM A.8. *Let $C(o,1)$ and $C(O,R)$ be perpendicular and $C(o,1)\cap C(O,R) = \{P,Q\}$. Take $A' \in PQ$ and let $A = \overline{oA'} \cap C(O,R)$. Then*

$$oA' = \frac{2oA}{1+oA'^2}.$$

PROOF. Let A'' be the inverse of A with respect to $C(o,1)$. Because the circles are perpendicular, we have $A'' \in c(O,R)$.

Denote $a = oA$, $a' = oA'$ and $a'' = oA''$. Let B and C be as in the picture. Since the circles have a common chord, we can apply Theorem A.7 to obtain

$$(1+a')(1-a') = BA'\cdot A'C = PA'\cdot A'Q = AA'\cdot A'A'' = (a'-a)(a''-a').$$

Because A'' be the inverse of A with respect to $C(o,1)$, we have $a'' = 1/a$. We obtain

$$1-(a')^2 = (a'-a)(1/a-a') = a'/a - 1 - (a')^2 + aa',$$

which yields

$$2 = \frac{a'}{a} + aa', \quad 2a = a' + a^2a', \quad a' = \frac{2a}{1+a^2}.$$

□

For $0 < t < 1$, denote $x = 2d_{\text{hyp}}(0,t)$ to have

$$e^x = e^{2d_{\text{hyp}}(0,t)} = \frac{1+t}{1-t}.$$

Now,

$$\sinh(x) = \frac{\frac{1+t}{1-t} + \frac{1-t}{1+t}}{2} = \frac{(1+t)^2-(1-t)^2}{2(1-t^2)} = \frac{2t}{1-t^2}.$$

Similarly,

$$\cosh(x) = \frac{1+t^2}{1-t^2}, \quad \tan(x) = \frac{2t}{1+t^2}.$$

Since

$$\int_0^r \frac{dt}{1-t^2} = \frac{1}{2}\int_0^s \frac{1}{1-t} + \frac{1}{1+t}dt = \frac{1}{2}\log\frac{1+r}{1-r} = \frac{1}{2}\operatorname{arctanh}(r),$$

the hyperbolic radius of an Euclidean disk satisfies

$$D_{d_E}(0,r) = D_{d_{\text{hyp}}}(0,\tanh r).$$

A.7.2. Hyperbolic right triangles.

THEOREM A.9. *Let $\triangle ABC$ be a hyperbolic triangle in $\mathbb{D}$ with angles α, β, γ and opposite sides a, b, c. Let $\gamma = 90°$. Then*
(i) $\cos\alpha = \frac{\tanh b}{\tanh c}$
(ii) $\sinh\alpha = \frac{\sinh a}{\sinh c}$

PROOF. (i) We may assume that $A = 0$. Let P, C, B, Q lie on a hyperbolic line and $P, Q \in \partial\mathbb{D}$. Let the circle containing $\underline{P, C, B, Q}$ be β and let its center be O_β. Draw PQ and let $B' = \overline{AB} \cap PQ$ $C' = \overline{AC} \cap PQ$ and be the projections of B and C on PQ.

By Theorem A.8, we have

$$\cos(\alpha) = \frac{OC'}{OB'} = \frac{2OC}{1+OC^2}\Big/\frac{2OB}{1+OB^2} = \frac{\tanh b}{\tanh c},$$

where for example (perhaps in a confusing way)

$$\tanh b = \frac{2OB}{1+OB^2}.$$

(ii) Let $\{B, B''\} = \overline{OB} \cap \beta$ and $\{C, C''\} = \overline{OC} \cap \beta$. The triangle $\triangle BO_\beta B''$ is isosceles and therefore midpoint B_1 of B and B'' satisfies $O_\beta B_1 \perp BB''$.

Since C'' is the inverse of C with respect to $\mathbb{D}$, we have

$$CC'' = OC'' - OC = \frac{1}{OC} - OC = \frac{1-OC^2}{OC} = \frac{2}{\sinh b}.$$

Similarly

$$BB'' = \frac{2}{\sinh c}.$$

Draw a tangent $\overline{BD}$ to β at B. By definition, $\beta = \angle ABC$ is the angle between $\overline{BD}$ and OB'', and we see that $\beta = \angle B_1 O_\beta B$. Hence

$$\sin\beta = \frac{2BB_1}{2BO_\beta} = \frac{BB''}{CC''} = \frac{\frac{2}{\sinh c}}{\frac{2}{\sinh b}} = \frac{\sinh b}{\sinh c}.$$

By a similar proof, we obtain

$$\sin\alpha = \frac{\sinh a}{\sinh c}.$$

□

Theorem A.10 is the hyperbolic Pythagoras theorem. We have the planar, hyperbolic and spherical Pythagoras' theorem[2]:

$$\begin{aligned} c^2 &= a^2 + b^2, \\ \cosh^2 c &= \cosh^2 a \cosh^2 b, \\ \cos(c) &= \cos(a)\cos(b). \end{aligned} \tag{A.12}$$

THEOREM A.10. *For the triangle in Theorem A.9, we have*

$$\cosh^2 c = \cosh^2 a \cosh^2 b.$$

PROOF. Since

$$1 = \cos^2\alpha + \sin^2\alpha = \frac{\tanh^2 b}{\tanh^2 c} + \frac{\sinh^2 a}{\sinh^2 c} = \frac{\sinh^2 b \cosh^2 c}{\sinh^2 c \cosh^2 b} + \frac{\sinh^2 a}{\sinh^2 c},$$

we obtain

$$\sinh^2 c \cosh^2 b = \sinh^2 b \cosh^2 c + \sinh^2 a \cosh^2 b.$$

Since $\sinh^2 c = \cosh^2 c - 1$ and $\sinh^2 a = \cosh^2 a - 1$, we obtain

$$\cosh^2 c \cosh^2 b - \cosh^2 b = \sinh^2 b \cosh^2 c + \cosh^2 a \cosh^2 b - \cosh^2 b.$$

Hence

$$\cosh^2 c = \cosh^2(\cosh^2 b - \sinh^2 b) = \cosh^2 a \cosh^2 b,$$

as desired. $\square$

A.7.3. Hyperbolic general triangles.

THEOREM A.11. *Let $\triangle ABC$ be a hyperbolic triangle with angles α, β, γ and opposite sides a, b, c. Then*

(i) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$.
(ii) $\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$
(iii) $\cosh c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}$.

PROOF. (i) Let $D \in \overline{AC}$ such that $BD \perp AC$. Assume that D is in between of A and C. Now

$$\begin{aligned} \cosh c &= \cosh AD \cosh BD \\ &= (\cosh b \cosh DC - \sinh b \sinh DC) \cosh BD \\ &= \cosh a \cosh b - \sinh b \sinh DC \frac{\cosh a}{\cosh DC} \\ &= \cosh a \cosh b - \sinh b \cosh a \tanh DC \\ &= \cosh a \cosh b - \sinh a \sinh b \frac{\tanh DC}{\tanh a} \\ &= \cosh a \cosh b - \sinh a \sinh b \cos\gamma. \end{aligned}$$

[2]If you like anagrams, these are of course Pythagoras, Hyptagoras and Sphytagora.

(ii) Now

$$\sin^2\alpha = 1-\cos^2\alpha = 1-\left(\frac{\cosh b\cosh c-\cosh a}{\sinh b\sinh c}\right).$$

Hence

$$\begin{aligned}\frac{\sinh^2\alpha}{\sinh^2 a} &= \frac{\sinh^2 b\sinh^2 c-\cosh^2 b\cosh^2 c+2\cosh a\cosh b\cosh c-\cosh^2 c}{(\sinh a\sinh b\sinh c)^2}\\ &= \frac{(1-\cosh^2 b)(1-\cosh^2 c)-\cosh^2 b\cosh^2 c+2\cosh a\cosh b\cosh c-\cosh^2 c}{(\sinh a\sinh b\sinh c)^2}\\ &= \frac{1-\cosh^2 a-\cosh^2 b-\cosh^2 c+2\cosh a\cosh b\cosh c}{(\sinh a\sinh b\sinh c)^2}.\end{aligned}$$

The obtained expression is symmetric with respect to a, b, c. This shows the claim.

(iii) We write

$$\begin{aligned}\cosh a\cosh b\cosh c &= \cosh a\cosh b(\cosh b_1\cosh b_2+\sinh b_1\sinh b_2)\\ &= \cosh c\cosh^2 d\cosh b_1\cosh b_2.\end{aligned}$$

We divide this by $\cosh b_1\cosh b_2$ to obtain

$$\cosh a\cosh b(1+\tanh b_1\tanh b_2) = \cosh c\cosh^2 d = \cosh c(1+\sinh^2 d).$$

We rearrange this to

$$\cosh a\cosh b-\cosh c = -\cosh a\cosh b\tanh b_1\tanh b_2+\cosh c\sinh^2 d.$$

By (i), we can replace the left hand side by $\sinh a\sinh b\cos\gamma$ to obtain

$$\sinh a\sinh b\cos\gamma = -\cosh a\cosh b\tanh b_1\tanh b_2+\cosh c\sinh^2 d.$$

Dividing by $\sinh a\sinh b$ yields

$$\begin{aligned}\cos\gamma &= -\frac{\tanh b_1}{\tanh b}\frac{\tanh b_2}{\tanh a}+\frac{\sinh d}{\sinh b}\frac{\sinh d}{\sinh a}\cosh b\\ &= -\cos\alpha\cos\beta+\sin\alpha\sin\beta\cosh c,\end{aligned}$$

which is the assertion. □

A.7.4. Areas of triangles.

THEOREM A.12. *The area A of a hyperbolic triangle with sides a, b, c satisfies*

$$\tan\frac{A}{2} = \tan\frac{a}{2}\tan\frac{b}{2}.$$

PROOF. We have

$$\cosh a = \frac{\cos\alpha}{\sin\beta},\quad \cosh b = \frac{\cos\beta}{\sin\alpha}$$

and therefore

$$\begin{aligned}\tan^2\frac{a}{2}\tan^2\frac{b}{2} &= \frac{\cosh a-1}{\cosh a+1}\frac{\cosh b-1}{\cosh b+1}\\ &= \frac{\cos\alpha-\sin\beta}{\cos\alpha+\sin\beta}\frac{\cos\beta-\sin\alpha}{\cos\beta+\sin\alpha} \qquad \text{(A.13)}\\ &= \frac{\cos(\alpha-\beta)-(\cos\alpha\sin\alpha+\sin\beta\cos\beta)}{\cos(\alpha-\beta)-(\cos\alpha\sin\alpha+\sin\beta\cos\beta)},\end{aligned}$$

since

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

Denoting for a moment

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=AB+ab,$$

we have

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=aB+bA$$

and multiplying these

$$\cos(\alpha-\beta)\sin(\alpha+\beta)=aAB^2+bBA^2+bBa^2+aAb^2=aA+bB=\sin\alpha\cos\alpha+\sin\beta\cos\beta.$$

We conclude

$$\frac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\cos(\alpha-\beta)}=\sin(\alpha+\beta).$$

Hence, by (A.13), we obtain

$$\tan^2\frac{a}{2}\tan^2\frac{b}{2}=\frac{1-\sin(\alpha+\beta)}{1+\sin(\alpha+\beta)}.$$

Now by

$$\sin(\alpha+\beta)=\cos(\pi/2-(\alpha+\beta))=\cos(A).$$

Hence

$$\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}=\frac{1-\sin(\alpha+\beta)}{1+\sin(\alpha+\beta)}=\tan^2\frac{a}{2}\tan^2\frac{b}{2},$$

which shows the claim. □

We needed the formula

$$\tan^2\frac{x}{2}=\left(i\tanh\frac{ix}{2}\right)^2=-\frac{\cosh(ix)-1}{\cosh(ix)+1}=\frac{1-\cos x}{1+\cos x}.$$

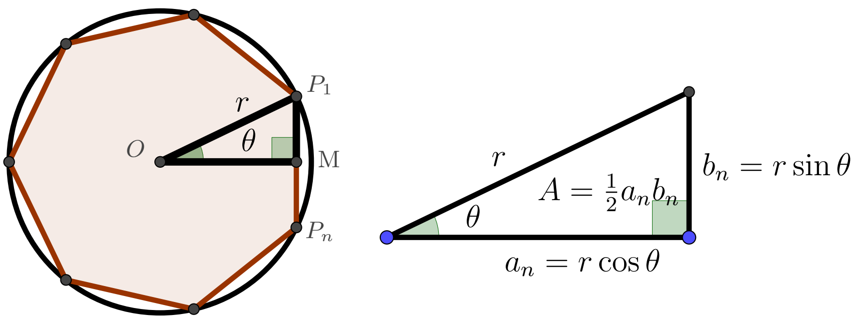


FIGURE A.1. Approximation of Euclidean disk by a polygon.

A.7.5. Area and circumference of a disk. An Euclidean triangle can be approximated by a polygon as Figure A.1 shows.

Assume that we are given a disc of radius r. Let $n \in \mathbb{N}$ with $n \geq 3$. Take n points $P_1, \dots, P_n$ from the circumference so that $\angle P_K O P_{k+1} = 2\pi/n$. Choose $M \in P_1P_n$ so that $OM \perp P_1P_N$ and denote $\theta = \angle MOP_1 = 2\pi/2n$. Then the circumference p_n of the polygon $P_1P_2 \cdots P_n$ satisfies

$$p_n = 2nb_n = 2nr\sin\theta = \frac{\sin\frac{2\pi}{2n}}{\frac{2\pi}{2n}}2\pi r \to 2\pi r$$

as $n \to \infty$. Similarly, the area of the polygon

$$A_n = 2n\frac{1}{2}a_nb_n = \frac{p_n}{2}a_n = \frac{p_n}{2}r\cos\frac{2\pi}{2n} \to \frac{2\pi r}{2}r = \pi r^2$$

as $n \to \infty$.

In conclusion, for an Euclidean circle the circumference is $2\pi r$ and the area is πr^2.

We will do the same approximation for a hyperbolic disk.

THEOREM A.13. *The hyperbolic circumference of a circle* $C_{d_{hyp}}(0, r)$ *is* $2\pi\sinh(r)$.

PROOF. Approximate the hyperbolic circumference of $C_{d_{\mathrm{hyp}}}(0, r)$ by an n-gon, whose circumference is p_n. Draw triangles with sides $b = p_n/2n$ and $a = r$, such that the angle at the chord is $\gamma = \pi/2$ and the angle at 0 is $\beta = \pi/n$. We obtain

$$\sinh b = \sinh a \sin\beta,$$

which reads

$$\sinh(p_n/2n) = \sinh r \sin(\pi/n).$$

Hence

$$p_n\frac{\sinh(p_n/2n)}{p_n/2n} = \sinh r\frac{\sin(\pi/n)}{\pi/n}2\pi.$$

Letting $n \to \infty$, we obtain

$$L(C(0,r)) = 2\pi \sinh r.$$

□

THEOREM A.14. *Area of a hyperbolic disc of hyperbolic radius r is*

$$A = 4\pi \sinh^2(r/2).$$

We note that, since

$$A = 4\pi \sinh^2(r/2) = \pi r^2 \left(\frac{\sinh(r/2)}{r/2}\right)^2 \sim \pi r^2$$

as $r \to 0$, the assertion seems reasonable for small discs.

PROOF. Approximate the disk by a n-sided polygon, whose area is K_n and perimeter is p_n. Divide the polygon to $2n$ right triangles with hypotenuse r, longer side a_n and shorter side $b_n = p_n/2n$. We obtain

$$\tan \frac{K_n/2n}{2} = \tanh \frac{p_n/2n}{2} \tanh \frac{a_n}{2}$$

and further

$$K_n \frac{\tanh \frac{K_n}{4n}}{\frac{K_n}{4n}} = \frac{\tanh \frac{p_n}{4n}}{p_n/4n} \tanh \frac{a_n}{2} p_n.$$

Taking a limit $n \to \infty$, we have $p_n \to 2\pi \sinh(r)$, and obtain

$$A = \left(\lim_{r\to\infty} p_n\right) \tanh\left(\lim_{n\to\infty} \frac{a_n}{2}\right) = 2\pi \sinh(r) \tanh \frac{r}{2}.$$

Since

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

we have

$$\sinh(2x)\tanh(x) = 2\sinh^2(x),$$

and we obtain

$$A = 4\pi \sinh^2 \frac{r}{2}.$$

□

On a sphere of radius R, the circumference of the spherical disc $D(0,r)$ is

$$2\pi R \sin \frac{r}{R} = 2\pi r \frac{\sin(r/R)}{r/R}.$$

If θ is the angle between the pole and the circle, this reads as

$$2\pi R \sin \theta.$$

By integrating from 0 to θ_0 against $dr = d(R\theta)$, we obtain

$$A = \int_0^{\theta_0} 2\pi R \sin\theta d(R\theta). = \left[2\pi R^2(-\cos\theta)\right]_{\theta=0}^{\theta_0} = 2\pi R^2(1-\cos\theta_0).$$

Hence, on a sphere of radius R, the area of the spherical disc $D(0,r)$ is

$$2\pi R^2(1-\cos(r/R)) = \pi r^2 \frac{1-\cos(r/R)}{(r/R)^2/2}.$$

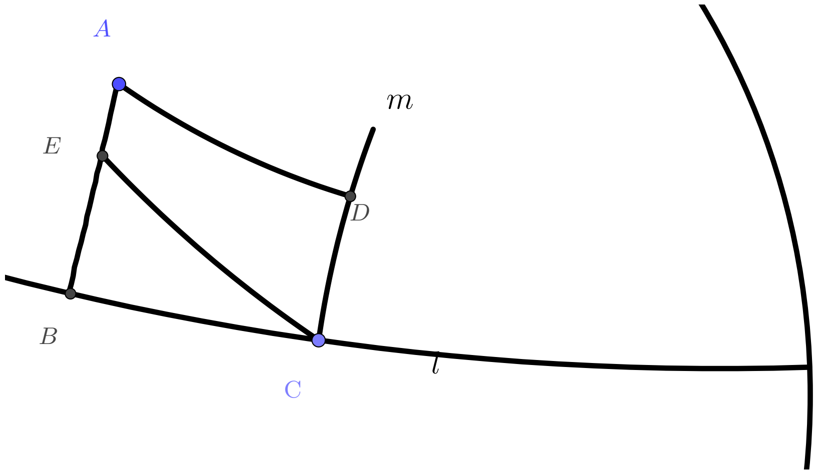


FIGURE A.2. Angle of parallelism.

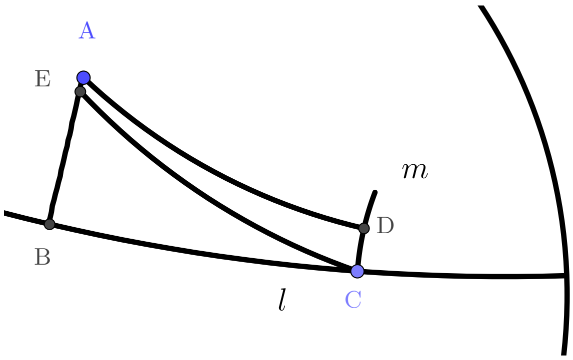


FIGURE A.3. Angle of parallelism.

A.7.6. Angle of parallelism. Let l be a hyperbolic line in $\mathbb{D}$ and $A \notin l$. Take $B \in l$ such that $AB \perp l$. Let $C \in l$ be different from B. Draw a line $m \perp l$ at C. Draw a perpendicular from A to $D \in m$, that is $AD \perp m$.

Now we have a quadrilateral $ABCD$ such that $\beta = \gamma = \delta = \pi/2$. However, $\alpha < \pi/2$ since the sum of angles is less than 2π for a hyperbolic quadrilateral. This is because the sum of angles is less than π for a hyperbolic triangle.

By the Pythagoras' theorem

$$\cosh AC = \cosh AD \cosh DC > \cosh AD \cdot 1 = \cosh AD$$

implying $AC > DA$.

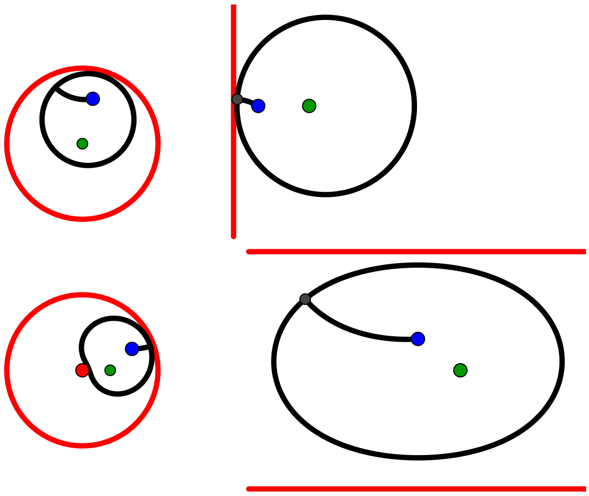


FIGURE A.4. Hyperbolic discs in different domains: $\mathbb{D}$, right half plane, strip $\left\{|\Im(z)| < \frac{\pi}{2}\right\}$ and $\mathbb{D} \setminus \{0\}$. Boundary in red, point corresponding to origin in green. Also a hyperbolic radius is shown.

By the Pythagoras' theorem

$$1 = \frac{\cosh AC}{\cosh AC} = \frac{\cosh AD \cosh DC}{\cosh BC \cosh AB}.$$

Now, we have three options:

(1) $BC < AD$ and $AB < CD$,
(2) $BC = AD$ and $AB = CD$,
(3) $BC > AD$ and $AB < CD$.

For example, if $BC > AD$ and $AB > CD$, then

$$\frac{\cosh AD \cosh DC}{\cosh BC \cosh AB} < \frac{\cosh BC \cosh AB}{\cosh BC \cosh AB} = 1,$$

which is a contradiction. Similarly, $BC > AD$ and $AB > CD$ cannot hold at the same time.

Assume that (1) does not hold. Since $BC \geq AD$ and $AB \leq CD$,

$$\sin \angle BAC = \frac{\sinh BC}{\sinh AC} \geq \frac{\sinh AD}{\sinh AC} \sin \angle DCA$$

and

$$\sin \angle CAD \frac{\sinh CD}{\sinh AC} \geq \frac{\sinh AB}{\sinh AC} = \sin \angle ACB.$$

This leads to

$$\alpha = \angle BAD = \angle BAC + \angle CAD \geq \angle DCA + \angle ACB = \angle DCB = \pi/2,$$

which is a contradiction. Hence (1) holds.

In conclusion $BC < DA < AC$. Draw the circle $C(C, DA)$. Since $BC < DA$, this circle woult intersect BC. Since $DA < AC$, the circle will intersect AB. Let $E = C(C, DA) \cap AB$.

We study the angle $\angle BEC$. We have

(A.14)
$$\begin{aligned}\sin BEC &= \frac{\sinh BC}{\sinh EC} = \frac{\sinh BC}{\sinh DA} = \frac{\sinh BC}{\sin\angle ACD \sinh CA} \\ &= \frac{\sinh BC}{\cos ACB \sinh CA} = \frac{\sinh BC \tanh CA}{\tanh CB \sinh CA} = \frac{\cosh BC}{\cosh CA} \\ &= \frac{\cosh BC}{\cosh CB \cosh AB} = \frac{1}{\cosh AB}.\end{aligned}$$

We find the remarkable fact that $\angle BEC$ is independent of the choice of C.

The point C can be moved to infinity and $\angle BEC$ will not change.

It is seen that if and only if

$$\angle BAF < \angle BAC = \operatorname{arcsin}\left(\frac{1}{\cosh AB}\right),$$

then AF intersects $\overline{BC}$.

The angle

$$\angle BAC = \operatorname{arcsin}\left(\frac{1}{\cosh AB}\right)$$

is the so-called angle of parallelism.

A.8. Equilateral triangles forming regular n-gons

A.8.1. The plane. In the plane, six equilateral triangles of any size make a regular hexagon.

This is because a related right triangle has an angle satisfying

$$\sin\frac{\alpha}{2} = \frac{r/2}{r} = \frac{1}{2},$$

and hence

$$\alpha = 2\operatorname{arcsin}(1/2) = \frac{2\pi}{6}$$

for any $r \in (0, \infty)$.

A.8.2. On the sphere. On the sphere, equilateral triangles can form a regular n-gon for $n \in \{2, 3, 4, 5\}$. However, for $n = 2$, the n-gon is degenerate: the closure of its complement is the whole sphere.

Moreover, the regular n-gons can be used to tesselate the sphere. The convex hull of the vertices is a Platonic solid.

We wish to calculate as in the plane case. The spherical law of sines reads

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b}$$

the corresponding right triangle satisfies

$$\frac{\sin(\alpha/2)}{\sin(r/2)} = \frac{\sin 90°}{\sin r} = \frac{1}{\sin r}$$

implying

$$\sin\frac{\alpha}{2} = \frac{\sin(r/2)}{\sin r} = \frac{\sin(r/2)}{2\sin(r/2)\cos(r/2)} = \frac{1}{2\cos(r/2)}.$$

If the sphere has radius $R = 1$, then $r = R\theta = \theta$ and we obtain

$$\theta = \frac{r}{R} = 2\arccos\left(\frac{1}{2\sin(\alpha/2)}\right) = 2\arccos\left(\frac{1}{2\sin(\pi/n)}\right).$$

If $n = 2$, we obtain by $\sin(\pi/2) = 1$ that

$$\theta = 2\arccos\frac{1}{2} = 2\frac{\pi}{3} = \frac{2\pi}{3}.$$

The Platonic solid is degenerated and is actually a equilateral triangle.

If $n = 3$, we obtain by $\sin(\pi/3) = \sqrt{3}/2$ that

$$\theta = 2\arccos\frac{1}{\sqrt{3}} \approx 109,5^\circ.$$

The Platonic solid is a tetrahedron.

If $n = 4$, we obtain by $\sin(\pi/4) = 1/\sqrt{2}$ that

$$\theta = 2\arccos\frac{1}{\sqrt{2}} = 2\frac{\pi}{4} = \frac{\pi}{2}.$$

The Platonic solid is an octagon.

For the case $n = 5$, we note that

$$\sin(\pi/5) = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}.$$

and hence

$$\frac{1}{2\sin(\pi/5)} = \sqrt{\frac{2}{5-\sqrt{5}}} = \sqrt{\frac{5+\sqrt{5}}{10}}.$$

Therefore

$$\theta = 2\arccos\sqrt{\frac{5+\sqrt{5}}{10}} \approx 63,43^\circ.$$

The Platonic solid is an icosahedron.

A.8.3. On the hyperbolic plane. We will find that equilateral triangles can form an n-gon for $n \geq 7$.

The right triangle satisfies

$$\sin\frac{\alpha}{2} = \frac{\sinh(r/2)}{\sinh(r)} = \frac{1}{\cosh(r/2)}.$$

Hence

$$r = 2\,\mathrm{arcCosh}\,\frac{1}{2\sin(\alpha/2)} = 2\,\mathrm{arcCosh}\,\frac{1}{2\sin(\pi/n)}.$$

For large n, $\sin(\pi/n) \approx \pi/n$ and we obtain

$$r \approx \mathrm{arcCosh}\,\frac{n}{2\pi}.$$

For large x, $\cosh x \approx e^2/2$, $\operatorname{arcCosh} x \approx \log(2x)$ and we obtain

$$r(n) \approx \log \frac{\pi}{n}.$$

Next, we list a few values $r(n)$.

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