# ON A REVERSAL OF THE SECOND MAIN THEOREM FOR MEROMORPHIC FUNCTIONS OF FINITE ORDER

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## 1. Introduction

Let $f$ be a non-constant meromorphic function in the plane. We set

$$\bar{m}_q(r,f) = \sup_{(a_1,\cdots,a_q)\in(\hat{\mathbb{C}})^q} \int_0^{2\pi} \max_{1\leq i\leq q} \log \frac{1}{[f(re^{i\theta}),a_i]} \frac{d\theta}{2\pi}.$$

Here $[x,y]$ is the chordal distance between two points in the extended complex plane:

$$[x,y] = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

We prove the following theorem.

**Theorem 1.** *Let $f$ be a transcendental meromorphic function of finite order. Let $\nu : \mathbb{R}_{>e} \to \mathbb{N}_{>0}$ satisfies $\nu(r) \to \infty$ and $\log \nu(r) = o(T(r,f))$ as $r \to \infty$. Then we have*

$$\bar{m}_{\nu(r)}(r,f) + N_1(r,f) = 2T(r,f) + o(T(r,f)) \tag{1.1}$$

*where $r \to \infty$ outside a set of logarithmic density* 0.

In [8, Theorem 1.6], the estimate (1.1) is proved for general transcendental meromorphic functions, including the case of infinite order, provided that the function $\nu$ satisfies

$$\nu(r) \sim \left(\log^+ \frac{T(r,f)}{\log r}\right)^{20}. \tag{1.2}$$

Our theorem shows that $\nu(r)$ may be arbitrary slow growth if $f$ is of finite order.

The proof of Theorem 1 is quite similar to that of [8, Theorem 1.6]. If $\nu : \mathbb{R}_{>e} \to \mathbb{N}_{>0}$ satisfies $\log \nu(r) = o(T(r,f))$, then a uniform version of Nevanlinna's second main theorem yields

$$\bar{m}_{\nu(r)}(r,f) + N_1(r,f) \leq 2T(r,f) + o(T(r,f)) \tag{1.3}$$

for all $r > e$ outside an exceptional set of finite linear measure (cf. [8, Section 1.6]). Thus the issue is to prove the reversal of (1.3). This is contained in the following theorem.

**Theorem 2.** *Let $f$ be a transcendental meromorphic function of finite order $\lambda$. For $0 < \varepsilon < 1$, there exist a positive integer $q_{\lambda,\varepsilon}$ and a set $E_{f,\varepsilon} \subset [e,\infty)$ with*

$$\overline{\log \operatorname{dens}}\, E_{f,\varepsilon} < \varepsilon$$

*such that for all $r \geq e$ outside $E_{f,\varepsilon}$, the following inequality holds:*

$$2T(r,f) \leq \bar{m}_{q_{\lambda,\varepsilon}}(r,f) + N_1(r,f) + \varepsilon T(r,f).$$

*Here $q_{\lambda,\varepsilon}$ only depends on $\lambda$ and $\varepsilon$.*

Here we denote by $\overline{\log \text{dens}}\, E$ the upper logarithmic density of E:

$$\overline{\log \text{dens}}\, E = \varlimsup_{r\to\infty} \frac{\int_{E\cap[e,r]} \frac{dt}{t}}{\log r}.$$

The proof of Theorem 2 shows that we may take $q_{\lambda,\varepsilon} = \lceil (2^{203} 2^{7680\lambda/\varepsilon^2})/\varepsilon^{20} \rceil$, where $\lceil x \rceil$ is the smallest integer which is not less than $x$.

*Remark.* Let $a_1, \dots, a_q \in \hat{\mathbb{C}}$ be distinct points. We have

$$\sum_{i=1}^{q} m(r, a_i, f) = \int_0^{2\pi} \max_{1\leq i \leq q} \log \frac{1}{[f(re^{i\theta}), a_i]} \frac{d\theta}{2\pi} + O(1) \leq \bar{m}_q(r, f) + O(1),$$

where $O(1)$ only depends on $a_1, \dots, a_q$. Thus we may recover usual estimate of Nevanlinna's second main theorem

$$\sum_{i=1}^{q} m(r, a_i, f) + N_1(r, f) \leq 2T(r, f) + o(T(r, f)) \tag{1.4}$$

from (1.3), provided $\nu(r) \to \infty$ as $r \to \infty$.

The question of reversal of (1.4) is already discussed in [5] and is a theme of [7, Chapter 4]. For many familiar functions, (1.4) is known to be an asymptotic equality rather than inequality. For instance, this holds for meromorphic functions with finitely many critical and asymptotic values, provided $\{a_1, \dots, a_q\}$ contains all critical and asymptotic values (cf. [6]). See also [2] for other investigation of this problem from potential-theoretic view point. Our quantity $\bar{m}$ is introduced in [8] to resolve conjectures of Mues and Gol'dberg concerning value distribution of derivatives of meromorphic functions.

## 2. Notations of Nevanlinna Theory

Let $f$ be a non-constant meromorphic function in the complex plane. Put $\mathbb{C}(t) = \{z \in \mathbb{C} : |z| < t\}$. We denote by $T(r, f)$ the spherical characteristic function of $f$, i.e.,

$$T(r, f) = \int_1^r \left( \int_{\mathbb{C}(t)} f^* \omega_{\hat{\mathbb{C}}} \right) \frac{dt}{t},$$

where

$$\omega_{\hat{\mathbb{C}}} = \frac{1}{(1 + |w|^2)^2} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w}$$

is the Fubini-Study form on the Riemann sphere $\hat{\mathbb{C}}$.

We denote by $n_1(t, f)$ the number of critical points of $f$ in $\mathbb{C}(t)$, counting multiplicity. We define the ramification counting function $N_1(r, f)$ by

$$N_1(r, f) = \int_1^r n_1(t, f) \frac{dt}{t}.$$

Let $a \in \hat{\mathbb{C}}$. We define the proximity function $m(r, a, f)$ by

$$m(r, a, f) = \int_0^{2\pi} \log \frac{1}{[f(re^{i\theta}), a]} \frac{d\theta}{2\pi}.$$

The detail of Nevanlinna theory may be found in [1], [3], [4], [5], [9].

## 3. Proof of the theorems

For a meromorphic function $f$, we put

$$v(r,f,\theta)=\sup_{\tau}\left(\sup_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|-\inf_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|\right).$$

We first show

**Proposition 1.** *Let $f$ be a transcendental meromorphic function of finite order $\lambda$. Let $0<\varepsilon<1$. Then there exists a positive constant $\theta_{\lambda,\varepsilon}$ such that*

$$v(r,f,\theta_{\lambda,\varepsilon})\le\varepsilon T(r,f)$$

*for all $r>e$ outside an exceptional set $E_{f,\varepsilon}$ with $\overline{\log\operatorname{dens}}\,E_{f,\varepsilon}<\varepsilon$.*

The proof of Proposition 1 shows that we may take $\theta_{\lambda,\varepsilon}=\varepsilon^{20}/2^{140}2^{120\lambda/\varepsilon^2}$. To prove Proposition 1, we need several lemmas.

**Lemma 1.** *For $0<\varepsilon<1$, there exists $\tau_\varepsilon>0$ such that*

$$\int_r^{2r}\frac{v(t,f,\tau_\varepsilon)}{t}dt<\varepsilon T(8r,f)$$

*for $r>r_0$, where $r_0>1$ is a constant which only depends on $f$.*

The proof shows that we may take $\tau_\varepsilon=\varepsilon^{10}/2^{110}$.

*Proof.* By [8, Lemma 3.2], we have the following: Let $1<\sigma<e$. Then

$$\int_r^{\sigma r}\frac{v(t,f,(\log\sigma)^{10})}{t}dt<508(\log\sigma)^2(T(\sigma^3r,f)+c) \tag{3.1}$$

for $r>1$, where $c$ is a positive constant which only depends on $f$.

Now given $0<\varepsilon<1$, we take a positive integer $l$ such that

$$l\ge\frac{1016(\log 2)^2}{\varepsilon}.$$

We take $r_0>1$ such that $T(r_0,f)>c$. Then for $i=0,\dots,l-1$ and $r>r_0$, (3.1) yields

$$\int_{2^{i/l}r}^{2^{(i+1)/l}r}\frac{v(t,f,(\log 2^{1/l})^{10})}{t}dt<1016(\log 2^{1/l})^2T(2^{(3+i)/l}r,f)\le\frac{1016(\log 2)^2}{l^2}T(8r,f).$$

Thus we get

$$\int_r^{2r}\frac{v(t,f,(\log 2^{1/l})^{10})}{t}dt<\varepsilon T(8r,f)$$

for $r>r_0$. We set $\tau_\varepsilon=(\log 2^{1/l})^{10}$ to conclude the proof. □

In order to deal with the term $T(8r,f)$, we need a growth lemma.

**Lemma 2.** *Let $g(r)$ be a continuous, non-decreasing function in $[e,\infty)$ and $g(e)>0$. Suppose that*

$$M=\varlimsup_{r\to\infty}\frac{\log g(r)}{\log r}<\infty.$$

*Given $0<\varepsilon<1$, put*

$$C(\varepsilon)=2\cdot 8^{2M/\varepsilon},$$
$$E_\varepsilon=\{r\in[e,\infty);g(8r)\ge C(\varepsilon)g(r)\}.$$

*Then we have*

$$\overline{\log\ \mathrm{dens}}\, E_\varepsilon < \varepsilon.$$

*Proof.* Suppose that $E_\varepsilon$ is bounded, then our lemma is trivial. Thus in the following, we assume that $E_\varepsilon$ is not bounded.

We define a sequence of positive numbers $r_1, r_2, \cdots$ by the following inductive rule:

$$r_1 = \inf E_\varepsilon,$$

$$r_{i+1} = \inf\left(E_\varepsilon \cap [8r_i, \infty)\right).$$

Since $E_\varepsilon$ is a closed set, we have $r_i \in E_\varepsilon$. Hence we have

$$g(r_{i+1}) \geq g(8r_i) \geq C(\varepsilon) g(r_i). \tag{3.2}$$

Now given large $R$ with $E_\varepsilon \cap [e, R] \neq \emptyset$, there is a positive integer $n(R)$ such that

$$E_\varepsilon \cap [e, R] \subset \bigcup_{i=1}^{n(R)} [r_i, 8r_i]$$

and

$$r_{n(R)} \leq R.$$

Then since

$$\int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} \leq \sum_{i=1}^{n(R)} \int_{r_i}^{8r_i} \frac{dt}{t} \leq n(R) \log 8,$$

we have

$$n(R) \geq \frac{1}{\log 8} \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t}.$$

Hence by (3.2), we have

$$\begin{aligned}
\log g(R) \geq \log g(r_{n(R)}) &\geq \log\left(C(\varepsilon)^{n(R)-1} g(r_1)\right) \\
&= n(R) \log C(\varepsilon) - \log C(\varepsilon) + \log g(r_1) \\
&\geq \left(\frac{1}{3} + \frac{2M}{\varepsilon}\right) \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} - \log C(\varepsilon) + \log g(r_1).
\end{aligned}$$

Hence we have

$$\varlimsup_{R\to\infty} \frac{1}{\log R} \int_{E_\varepsilon \cap [e,R]} \frac{dt}{t} \leq \left(\frac{3\varepsilon}{6M + \varepsilon}\right) \varlimsup_{R\to\infty} \frac{\log g(R) + \log C(\varepsilon) - \log g(r_1)}{\log R} < \varepsilon.$$

This proves our lemma. $\square$

**Lemma 3.** *Let $F \subset \mathbb{R}_{>e}$ be a measurable set, and let $\alpha \geq 0$. We define a set $E$ by*

$$E = \left\{ r;\ \int_{F \cap [r,2r]} \frac{dt}{t} > \alpha \right\}.$$

*Then we have*

$$\overline{\log\ \mathrm{dens}}\, F \leq \frac{\alpha}{\log 2} + \overline{\log\ \mathrm{dens}}\, E.$$

*Proof.* Put $G = [e, \infty)\backslash E$. Then $G$ is a closed set. Suppose that $G$ is bounded. In this case, the upper logarithmic density of $E$ is equal to 1, so our lemma is trivial. Hence in the following, we assume that $G$ is unbounded.

We define a sequence of positive numbers $\{r_n\}$ by the following inductive rule:

$$r_0 = e,$$

$$r_{i+1} = \begin{cases} 2r_i & r_i \in G \\ \inf[r_i, \infty) \cap G & r_i \notin G \end{cases}$$

Since we are assuming that $G$ is unbounded, this sequence is infinite. We observe that

$$r_{i+2} \geq 2r_i. \tag{3.3}$$

Indeed, this is obvious if $r_i \in G$. Suppose that $r_i \notin G$. Then since $G$ is closed, we conclude $r_{i+1} \in G$. Hence $r_{i+2} = 2r_{i+1}$, and we conclude (3.3) for $r_i \notin G$. From (3.3), we see that the sequence $\{r_n\}$ tends to infinity.

Now given $R > e$, there is a non-negative integer $n(R)$ such that

$$r_{n(R)} \leq R < r_{n(R)+1}.$$

We put

$$A = \{i \in \mathbb{Z}_{\geq 0};\ r_i \in G \text{ and } i \leq n(R) - 1\},$$
$$B = \{i \in \mathbb{Z}_{\geq 0};\ r_i \notin G \text{ and } i \leq n(R) - 1\}.$$

Then for the cardinarity of $A$, we have

$$|A| \leq \frac{\log(R/e)}{\log 2}.$$

Hence we have

$$\begin{aligned}
\int_{[e,R]\cap F} \frac{dt}{t} &= \sum_{i=0}^{n(R)-1} \int_{[r_i, r_{i+1}]\cap F} \frac{dt}{t} + \int_{[r_{n(R)}, R]\cap F} \frac{dt}{t} \\
&= \sum_{i\in A} \int_{[r_i, r_{i+1}]\cap F} \frac{dt}{t} + \sum_{i\in B} \int_{[r_i, r_{i+1}]\cap F} \frac{dt}{t} + \int_{[r_{n(R)}, R]\cap F} \frac{dt}{t} \\
&\leq \alpha(|A| + 1) + \int_{[e,R]\cap E} \frac{dt}{t} \\
&\leq \alpha\left(\frac{\log(R/e)}{\log 2} + 1\right) + \int_{[e,R]\cap E} \frac{dt}{t}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\overline{\lim_{R\to\infty}} \frac{1}{\log R} \int_{[e,R]\cap F} \frac{dt}{t} &\leq \alpha \overline{\lim_{R\to\infty}} \left(\frac{1}{\log 2} + \frac{1}{\log R}\right) + \overline{\lim_{R\to\infty}} \frac{1}{\log R} \int_{[e,R]\cap E} \frac{dt}{t} \\
&\leq \frac{\alpha}{\log 2} + \overline{\log \text{dens}}\, E.
\end{aligned}$$

This proves our lemma. $\square$

*Proof of Proposition 1.* Let $0 < \varepsilon < 1$. First we apply Lemma 1 for

$$\frac{\varepsilon^2/4}{C(\varepsilon^2/2)},$$

where $C(\varepsilon^2/2) = 2 \cdot 8^{4\lambda/\varepsilon^2}$ is the constant from Lemma 2. Then we get a positive constant $\theta_{\lambda,\varepsilon}$ such that

$$\int_r^{2r} \frac{v(t,f,\theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon^2/4}{C(\varepsilon^2/2)} T(8r,f)$$

for $r > r_0$. Here $\theta_{\lambda,\varepsilon} = \tau_{\varepsilon^2/2^{3+(12\lambda/\varepsilon^2)}}$.

Next we apply Lemma 2 for $\varepsilon^2/2$ to get a set $E$ such that

$$T(8r,f) < C(\varepsilon^2/2)T(r,f)$$

for all $r$ outside $E$. Here we have

$$\overline{\log\text{ dens}}\, E < \frac{\varepsilon^2}{2}.$$

Thus we have

$$\int_r^{2r} \frac{v(t,f,\theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon^2}{4} T(r,f)$$

for all $r > r_0$ outside $E$.

Now we set

$$F = \{r;\ v(r,f,\theta_{\lambda,\varepsilon}) \geq \varepsilon T(r,f)\}.$$

Then we have

$$\int_{[r,2r]\cap F} \frac{dt}{t} \leq \int_r^{2r} \frac{v(t,f,\theta_{\lambda,\varepsilon})}{\varepsilon T(t,f)t} dt \leq \frac{1}{\varepsilon T(r,f)} \int_r^{2r} \frac{v(t,f,\theta_{\lambda,\varepsilon})}{t} dt < \frac{\varepsilon}{4}$$

for all $r > r_0$ outside $E$. Thus by Lemma 3, we have

$$\overline{\log\text{ dens}}\, F < \frac{\varepsilon}{4\log 2} + \frac{\varepsilon^2}{2} < \varepsilon.$$

We conclude the proof of Proposition 1. □

*Now we prove Theorem 2.* Let $q > 0$ be a positive integer. We claim

$$2T(r,f) \leq \bar{m}_q(r,f) + N_1(r,f) + 2v(r,f,2\pi/q) + v(r,f',2\pi/q) + \log r + C \tag{3.4}$$

for all $r > 1$, where $C$ is a positive constant which only depends on $f$. This is a consequence of more general results given in Lemmas 3.6 and 3.7 in [8]. However we shall give a direct proof of (3.4) in the following, for the direct proof is simpler than the general one.

Let $\sigma_k = 2\pi k/q$. For $l = 0, 1, \cdots, q-1$, we set $I_l = [\sigma_l, \sigma_{l+1}]$ and $a_l = f(re^{i\sigma_l})$. We have

$$[f(re^{i\theta}), a_l] \leq \int_{\sigma_l}^{\theta} f^{\#}(re^{i\theta})\ rd\theta,$$

where $f^{\#}$ is the spherical derivative defined by

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

Set $\tau_l = \max_{s\in I_l} \log f^{\#}(re^{is})$. Then for $\theta \in I_l$, we have

$$[f(re^{i\theta}), a_l] \leq e^{\tau_l} 2\pi r/q. \tag{3.5}$$

We set

$$v(r,f^{\#},\theta) = \sup_{\tau} \left( \sup_{t\in[\tau,\tau+\theta]} \log f^{\#}(re^{it}) - \inf_{t\in[\tau,\tau+\theta]} \log f^{\#}(re^{it}) \right).$$

Then for $\theta \in I_l$, we have

$$\log \frac{1}{f^{\#}(re^{i\theta})} \le -\tau_l + v(r, f^{\#}, 2\pi/q).$$

Combining this estimate with (3.5), we get

$$\log \frac{1}{f^{\#}(re^{i\theta})} \le \log \frac{1}{[f(re^{i\theta}), a_l]} + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q)$$

for $\theta \in I_l$. Thus

$$\int_0^{2\pi} \log \frac{1}{f^{\#}(re^{i\theta})} \frac{d\theta}{2\pi} \le \sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f(re^{i\theta}), a_l]} \frac{d\theta}{2\pi} + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q).$$

By

$$\sum_{l=0}^{q-1} \int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f(re^{i\theta}), a_l]} \frac{d\theta}{2\pi} \le \bar{m}_q(r, f),$$

we conclude

$$\int_0^{2\pi} \log \frac{1}{f^{\#}(re^{i\theta})} \frac{d\theta}{2\pi} \le \bar{m}_q(r, f) + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q).$$

Combining this with the following well-known estimate (cf. [1, Proposition 2.4.2])

$$\int_0^{2\pi} \log f^{\#}(re^{i\theta}) \frac{d\theta}{2\pi} = -2T(r, f) + N_1(r, f) + \int_0^{2\pi} \log f^{\#}(e^{i\theta}) \frac{d\theta}{2\pi},$$

we get

$$2T(r, f) - N_1(r, f) \le \bar{m}_q(r, f) + v(r, f^{\#}, 2\pi/q) + \log(2\pi r/q) + \int_0^{2\pi} \log f^{\#}(e^{i\theta}) \frac{d\theta}{2\pi}.$$

By

$$v(r, f^{\#}, 2\pi/q) \le 2v(r, f, 2\pi/q) + v(r, f', 2\pi/q),$$

we conclude (3.4).

Now let $0 < \varepsilon < 1$. Set $q_{\lambda,\varepsilon} = \lceil 2\pi/\theta_{\lambda,\varepsilon/8} \rceil$. By Proposition 1, we have

$$v(r, f, 2\pi/q_{\lambda,\varepsilon}) < \frac{\varepsilon}{8} T(r, f) \tag{3.6}$$

for all $r > e$ outside $E_1$ with

$$\overline{\text{log dens}}\, E_1 < \frac{\varepsilon}{8}. \tag{3.7}$$

Since $f'$ has the same order $\lambda$, Proposition 1 yields that

$$v(r, f', 2\pi/q_{\lambda,\varepsilon}) < \frac{\varepsilon}{8} T(r, f')$$

for all $r > e$ outside $E_2$ with

$$\overline{\text{log dens}}\, E_2 < \frac{\varepsilon}{8}.$$

By Nevanlinna's Lemma on logarithmic derivative, we have

$$T(r, f') \le \frac{5}{2} T(r, f)$$

for all $r > e$ outside $E_3$ of finite linear measure. Hence we obtain

$$v(r, f', 2\pi/q_{\lambda,\varepsilon}) < \frac{5\varepsilon}{16}T(r,f) \tag{3.8}$$

for $r > e$ and $r \notin E_2 \cup E_3$, where we have

$$\overline{\log\,\mathrm{dens}}(E_2 \cup E_3) < \frac{\varepsilon}{8}. \tag{3.9}$$

Since $f$ is transcendental, we find a positive constant $r_1$ such that

$$\log r + C < \frac{\varepsilon}{8}T(r,f) \tag{3.10}$$

for $r > r_1$.

Now we put

$$E = [e, r_1] \cup E_1 \cup E_2 \cup E_3.$$

Then by (3.7) and (3.9), we have

$$\overline{\log\,\mathrm{dens}}\,E < \varepsilon.$$

By (3.6), (3.8), (3.10), we have

$$2v(r, f, 2\pi/q_{\lambda,\varepsilon}) + v(r, f', 2\pi/q_{\lambda,\varepsilon}) + \log r + C < \varepsilon T(r,f)$$

for all $r > e$ outside $E$. Combining this estimate with (3.4), we conclude the proof of Theorem 2.

*We prove Theorem 1.* Let $n$ be a positive integer. We recall $E_{f,1/2^n}$ and $q_{\lambda,1/2^n}$ from Theorem 2. By $\nu(r) \to \infty$ as $r \to \infty$, we may take $c_n > e$ such that $\nu(r) > q_{\lambda,1/2^n}$ for all $r > c_n$. We define $F_{1/2^n} \subset [e, \infty)$ such that $r \in F_{1/2^n}$ iff

$$2T(r,f) > \bar{m}_{\nu(r)}(r,f) + N_1(r,f) + \frac{1}{2^n}T(r,f).$$

Then we have $F_{1/2^n} \cap [c_n, \infty) \subset E_{f,1/2^n} \cap [c_n, \infty)$. Thus we have $\overline{\log\,\mathrm{dens}}\,F_{1/2^n} < 1/2^n$.

Now we take $r_n > e$ such that

$$\frac{\int_{F_{1/2^n}\cap[e,r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n},$$

for all $r \geq r_n$. We may assume without loss of generality that the sequence $r_1, r_2, \ldots$ satisfies $r_1 < r_2 < r_3 < \cdots$ and $r_n \to \infty$ as $n \to \infty$. For $r \in [r_n, r_{n+1})$, we set $\varepsilon(r) = 1/2^n$. Then $\varepsilon(r)$ is defined for all $r \geq r_1$ and $\varepsilon(r) \to 0$ as $r \to \infty$.

We define $F \subset [r_1, \infty)$ such that $r \in F$ iff

$$2T(r,f) > \bar{m}_{\nu(r)}(r,f) + N_1(r,f) + \varepsilon(r)T(r,f).$$

Then we have $F \cap [r_1, r_{n+1}) \subset F_{1/2^n} \cap [r_1, r_{n+1})$. Thus we have

$$\frac{\int_{F\cap[r_1,r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n}$$

for $r_n \leq r < r_{n+1}$. Thus $F$ has logarithmic density 0. Thus we have

$$2T(r,f) \leq \bar{m}_{\nu(r)}(r,f) + N_1(r,f) + o(T(r,f))$$

where $r \to \infty$ outside a set of logarithmic density 0. Combining this estimate with (1.3), we conclude the proof of Theorem 1.

## References


[1] W. Cherry and Z. Ye, *Nevanlinna's theory of value distribution. The second main theorem and its error terms*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.

[2] A. Eremenko and M. Sodin, *On the distribution of values of meromorphic functions of finite order*, Soviet Math. Dokl. **43** no. 1, (1991), 128–131.

[3] A. Goldberg and I. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 2008.

[4] W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.

[5] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag 1936. English translation: *Analytic Functions*, Springer-Verlag, 1970.

[6] O. Teichmüller, *Eine umkehrung des zweiten hauptsatzes der wertverteilungslehre*, Deutsche Mathematik, **2** (1937) 96–107.

[7] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 8. Springer-Verlag, Berlin-Gttingen-Heidelberg, 1955.

[8] K. Yamanoi, *Zeros of higher derivatives of meromorphic functions in the complex planes*, preprint.

[9] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin; Science Press, Beijing, 1993.



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