Research Seminar in Mathematics 2019
Exercise session: 22.3. at 10-12 in room M305

Exercise 1.1 Show that composition by a rotation of the Riemann sphere keeps the spherical derivative invariant. That is, $(g_j \circ f)^{\#}(z) = f^{\#}(z)$ for

(a) $g_1(z) = \alpha z$, where $|\alpha| = 1$,

(b) $g_2(z) = 1/z$,

(c) $g_3(z) = \dfrac{1 - \frac{1}{a}z}{z + \frac{1}{\bar{a}}}$, where $a \in \mathbb{C} \setminus \{0\}$. Solution.

Exercise 1.2 Show that the chordal metric[1] satisfies

(a) $[z, w] = [1/z, 1/w]$,

(b) $\dfrac{|c|}{1 + |c|^2}[z, w] \leq [cz, cw] \leq \dfrac{1 + |c|^2}{|c|}[z, w]$, for $z, w, c \in \mathbb{C}$. Solution.

Exercise 1.3 Let $a_j = p_j/q_j \in \mathcal{R}_d$, where p_j, q_j are polynomials of degree at most d. Show that there is a subsequence $(a_n)_{n\in\mathbb{N}}$ and a rational function $a \in \mathcal{R}_d$ such that $a_n \to a$ locally uniformly in the chordal metric outside a finite set of points in $\mathbb{C}$.

Hint. Method I. Use the theory of quasinormal families, see Schiff - Normal families pages 208–213.

Method II. Follow an elementary approach and show:
(i) If $b_k, c_k \in \mathbb{C}$, $k \in \mathbb{N}$, then some subsequence of $b_n z + c_n$, converges to a polynomial of degree one outside a finite set of points.
(ii) If a_j, b_j, c_j, d_j are sequences of complex numbers and

$$p_j(z) = a_j z^3 + b_j z^2 + c_j z + d_j = d_j + z(c_j + z(b_j + z a_j)),$$

then some subsequence p_n converges to a polynomial p of degree 3 outside a finite set of points.
(iii) Show the claim. Solution.

Exercise 1.4 Show that if φ is analytic and t is real, then

(i) $\frac{d}{dt}\overline{\varphi(t)} = \overline{\varphi'(t)}$,

(ii) $\frac{d}{dt}\mathrm{Re}(\varphi(t)) = \mathrm{Re}(\varphi'(t))$,

(iii) $\frac{d}{dt}|\varphi(t)|^2 = 2\,\mathrm{Re}(\varphi'(t)\overline{\varphi(t)})$. Solution.

[1] The chordal metric is connected to the spherical derivative by

$$f^{\#}(z) = \lim_{\zeta \to z} \frac{[f(z) - f(\zeta)]}{|z - \zeta|} = \frac{|f'(z)|}{1 + |f(z)|^2}, \quad f(z) \neq \infty.$$

Exercise 1.5 Let[2]

$$P(z,\theta)=\frac{1-|z|^2}{|e^{i\theta}-z|^2}=\mathrm{Re}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)$$

and

$$g(z,a)=\log\frac{1}{|\varphi_a(z)|}=-\mathrm{Re}\log\varphi_a(z),\quad \varphi_a(z)=\frac{a-z}{1-\bar{a}z}.$$

Show that

(i)

$$\frac{\partial}{\partial\theta}P(te^{i\theta},0)=-\frac{2t(1+t)(1-t)\sin\theta}{|1-te^{i\theta}|^4},$$

(ii) for $0<a<1$,

$$\frac{\partial g}{\partial\theta}(te^{i\theta},a)=\frac{at\sin\theta}{|1-ate^{i\theta}|^2}-\frac{at\sin\theta}{|a-te^{i\theta}|^2}.$$

Solution.

Exercise 1.6 Read the following part of the proof of Lemma 3.2. Check the details and include intermediate steps, where further explanation is required.

For $0<t<1$, let

$$\alpha(t,\theta,\tau)=\sup_{\tau_0\in[0,2\pi]}\left(\sup_{x\in[\tau_0,\tau_0+\tau]}P(te^{ix},\theta)-\inf_{x\in[\tau_0,\tau_0+\tau]}P(te^{ix},\theta)\right)$$

and

$$\beta(t,a,\tau)=\sup_{\tau_0\in[0,2\pi]}\left(\sup_{x\in[\tau_0,\tau_0+\tau]}g(te^{ix},a)-\inf_{x\in[\tau_0,\tau_0+\tau]}g(te^{ix},a)\right).$$

Set $\alpha(t,\tau)=\alpha(t,0,\tau)$. Since $P(z,\theta)=P(ze^{-i\theta},0)$, we have $\alpha(t,\theta,\tau)=\alpha(t,\tau)$.

For $1<\sigma<e$, let

$$\widetilde{\alpha}(\tau)=\int_{1/\sigma^2}^{1/\sigma}\alpha(t,\tau)\frac{dt}{t},\quad \widetilde{\beta}(\tau)=\sup_{|a|\le 1}\int_{1/\sigma^2}^{1/\sigma}\beta(t,a,\tau)\frac{dt}{t}.$$

Claim 1. We have

$$\widetilde{\alpha}(\tau)\le\frac{4e^2\tau}{\log\sigma}\le\frac{4\cdot 8\tau}{\log\sigma}.$$

Proof of Claim 1. Since (by Exercise 1.1(i))

$$\frac{\partial}{\partial\theta}P(te^{i\theta},0)=-\frac{2t(1+t)(1-t)\sin\theta}{|1-te^{i\theta}|^4},$$

and

$$|1-te^{i\theta}|=|1-t(\cos\theta+i\sin\theta)|=\sqrt{(1-t\cos\theta)^2+(t\sin\theta)^2)}\ge t|\sin\theta|,$$

we have

$$\left|\frac{\partial}{\partial\theta}P(te^{i\theta},0)\right|\le\frac{4}{(1-t)^2},\quad 0<t<1.$$

[2]Yamanoi has an extra factor $1/2\pi$ in the definition of $P(z,\theta)$. Because of this, the constants in Yamanoi's paper are slightly incorrect in this place.

Hence, we obtain (by the mean value theorem)

$$\alpha(t,\tau) \le \frac{4\tau}{(1-t)^2} \le \frac{4\tau}{(1-1/\sigma)^2} = \frac{4\tau\sigma^2}{(\sigma-1)^2}, \quad 0 < t < \frac{1}{\sigma}.$$

Thus, we have asdf

$$\widetilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t,\tau)\frac{dt}{t} \le \frac{4\tau\sigma^2}{(\sigma-1)^2}\log\sigma.$$

Since $\log\sigma \le \sigma - 1$ and[3] $\sigma^2 < e^2 < 8$, we complete the proof of the claim. □

Claim 2. Let $\delta = (\log\sigma)^2$. Then

$$\begin{aligned} \widetilde{\beta}(\tau) &\le \sigma^2\left(2\delta + 8\frac{\delta}{\log\sigma}\right) + \sigma\tau + \frac{\tau\log\sigma}{\delta} \\ &\le \frac{(e^2+1)\tau}{(\log\sigma)^7} + 10e^2(\log\sigma)^3 \le \frac{10\tau}{(\log\sigma)^7} + 90(\log\sigma)^3. \end{aligned} \tag{1.1}$$

Proof of Claim 2. We denote $\delta = (\log\sigma)^4$. For $|a| < 1$, we set

$$\big[1/\sigma^2, 1/\sigma\big] = I(a) \cup J(a),$$

where

$$\begin{aligned} I(a) &= \big[1/\sigma^2, 1/\sigma\big] \quad \cap \quad [\,|a| - \delta, |a| + \delta\,] \\ J(a) &= \big[1/\sigma^2, 1/\sigma\big] \quad \setminus \quad [\,|a| - \delta, |a| + \delta\,]. \end{aligned}$$

Then we have

$$\int_{1/\sigma^2}^{1/\sigma} \beta(t,a,\tau)\frac{dt}{t} \le \underbrace{\sigma^2}_{<e^2<9} \int_{I(a)} \beta(t,a,\tau)dt + \int_{J(a)} \beta(t,a,\tau)\frac{dt}{t}. \tag{1.2}$$

First, we estimate the first term on the right hand side of (1.2). Since

$$\left|\frac{1-\overline{a}z}{z-a}\right| = 1 + \frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2},$$

is maximised when $\arg(z) = \arg(a)$, we have

$$\beta(t,a,\tau) \le \max_{\theta\in[0,2\pi]} g(te^{i\theta}, a) = \log\frac{1-t|a|}{|t-|a||} \le \log\frac{1}{|t-|a||}.$$

We obtain

$$\int_{I(a)} \beta(t,a,\tau)dt \le 2\int_0^\delta \log\frac{1}{x}\,dx = 2(\log\sigma)^4 + 8(\log\sigma)^4\log\frac{1}{\log\sigma},$$

by $\int\log\frac{1}{x} = x + x\log\frac{1}{x}$. By $\log\frac{1}{x} \le \frac{1}{x}$ for $x \in (0,\infty)$ we have $\log(1/\log\sigma) \le (1/\log\sigma)$, and obtain

$$\begin{aligned} \int_{I(a)} \beta(t,a,\tau)\,dt &\le 2\delta + 8\frac{\delta}{\log\sigma} \\ &= 2(\log\sigma)^4 + 8(\log\sigma)^3 \le 10(\log\sigma)^3, \end{aligned} \tag{1.3}$$

since $1 < \sigma < e$.

Next we estimate the second term on the right-hand side of (1.2). Since

$$\beta(t,a,\tau) = \beta(t,|a|,\tau),$$

[3]recall: $1 < \sigma < e < 3$

it is enough to consider the case $0 < a < 1$. Since (by Exercise 1.5(ii))

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = \frac{at\sin\theta}{|1 - ate^{i\theta}|^2} - \frac{at\sin\theta}{|a - te^{i\theta}|^2},$$

and

$$|1 - \omega e^{i\theta}| \geq |\omega||\sin\theta|,$$

we have

$$\left|\frac{\partial g}{\partial \theta}(te^{i\theta}, a)\right| \leq \frac{1}{1 - at} + \frac{1}{a - t} \leq \frac{1}{1 - t} + \frac{1}{a - t}.$$

Hence, we obtain

$$\beta(t, a, \tau) \leq \frac{\tau}{1 - t} + \frac{\tau}{a - t}.$$

Hence, on $t \in J(a)$, we have

$$\begin{aligned} 1 - t &\geq 1/\sigma \\ a - t &\geq \delta = (\log\sigma)^4 \end{aligned}$$

and consequently

$$\beta(t, a, \tau) \leq \frac{\sigma\tau}{\sigma - 1} + \frac{\tau}{\delta}.$$

Since $\log\sigma \leq \sigma - 1$, we obtain

$$\beta(t, a, \tau) \leq \frac{\sigma\tau}{\log\sigma} + \frac{\tau}{\delta}.$$

Thus, we obtain

$$\int_{J(a)} \beta(t, a, \tau)\frac{dt}{t} \leq \sigma\tau + \frac{\tau\log\sigma}{\delta} \leq \frac{(e^2 + 1)\tau}{(\log\sigma)^7} \leq \frac{10\tau}{(\log\sigma)^7}. \tag{1.4}$$

From (1.2)–(1.4), we complete the proof of our claim. □

Hence

$$\widetilde{\alpha}(\tau) + \widetilde{\beta}(\tau) \leq \frac{4\sigma^2\tau}{\log\sigma} + \sigma^2\left(2\delta + 8\frac{\delta}{\log\sigma}\right) + \sigma\tau + \frac{\tau\log\sigma}{\delta},$$

where, for example $\delta = (\log\sigma)^4$.

Solutions

Solution 1.1. (a) Let $h(z) = g_1(f(z)) = \alpha f(z)$. Now

$$h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2} = \frac{|\alpha f'(z)|}{1+|\alpha f(z)|^2} = \frac{|f'(z)|}{1+|f(z)|^2} = f^{\#}(z).$$

(b) Let $h(z) = g_2(f(z)) = 1/f(z)$. Now

$$h'(z) = \left(\frac{1}{f(z)}\right)' = -\frac{f'(z)}{f(z)^2}$$

and

$$h^{\#}(z) = \frac{|f'(z)/f(z)^2|}{1+|1/f(z)|^2} = \frac{|f'(z)|}{1+|f(z)|^2} = f^{\#}(z).$$

(c) Let $h(z) = g_3(f(z))$, where

$$g_3(z) = \frac{1-\frac{z}{a}}{z+\frac{1}{\bar{a}}}.$$

Now

$$\begin{aligned} g_3'(z) &= \frac{(z+1/\bar{a})(1/a) - (1-z/a)}{(z+1/\bar{a})^2} \\ &= \frac{-\frac{z}{a} - \frac{1}{|a|^2} - 1 + \frac{z}{a}}{(z+1/\bar{a})^2} = -\frac{1+\frac{1}{|a|^2}}{(z+1/\bar{a})^2}. \end{aligned}$$

Since

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = |z|^2+|w|^2+z\bar{w}+\bar{z}w = |z|^2+|w|^2+2\operatorname{Re}(z\bar{w})$$

for all $z, w \in \mathbb{C}$, we have

$$\begin{aligned} 1+|g_3(z)|^2 &= \frac{|z+1/\bar{a}|^2+|1-z/a|^2}{|z+1/\bar{a}|^2} \\ &= \frac{|z|^2+\frac{1}{|a|^2}+2\operatorname{Re}\left(\frac{z}{a}\right)+1+\left|\frac{z}{a}\right|^2-2\operatorname{Re}\left(\frac{z}{a}\right)}{|z+1/\bar{a}|^2} \\ &= \frac{\left(1+\frac{1}{|a|^2}\right)(1+|z|^2)}{|z+1/\bar{a}|^2}. \end{aligned}$$

We see that

$$g_3^{\#}(z) = \frac{|g_3'(z)|}{1+|g_3(z)|^2} = \frac{1}{1+|z|^2}.$$

Hence for $h(z) = g_3(f(z))$, we obtain

$$h^{\#}(z) = \frac{|g_3'(f(z))||f'(z)|}{1+|g_3(f(z))|^2} = \frac{|f'(z)|}{1+|f(z)|^2} = f^{\#}(z).$$

The particular rotations

$$g_1(z) = \alpha z, \quad g_2(z) = \frac{1}{z}, \quad g_3(z) = \frac{1-\frac{1}{a}z}{z+\frac{1}{\bar{a}}},$$

satisfy that: g_1 keeps the origin fixed, g_2 exchanges 0 and ∞, and g_3 sends $a \neq 0, \infty$ to the origin. Clearly any rotation of the Riemann sphere is a composition of the mappings g_j.

Moreover, it can be shown that a Möbius transformation $T_S : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rotation of the Riemann sphere if and only if S is unitary. That is, T is of the form

$$T_s(z) = \lambda \frac{az+b}{\bar{a}-\bar{b}z}, \quad S = \lambda \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, and $|\lambda| = 1$.

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Solution 1.2. (a) We see that

$$\left[\frac{1}{z}, \frac{1}{w}\right] = \frac{\left|\frac{1}{z} - \frac{1}{w}\right| |zw|}{\sqrt{1+1/|z|^2}|z|\sqrt{1+1/|w|^2}|w|} = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} = [z,w]\,. \tag{1.5}$$

(b) Since $1 + |ca|^2 \le (1+|c|^2)(1+|a|^2)$, we obtain

$$\begin{aligned} [cz, cw] &= \frac{|cz-cw|}{\sqrt{1+|cz|^2}\sqrt{1+|cw|^2}} \\ &\ge \frac{|c|}{(\sqrt{1+|c|^2})^2} \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} = \frac{|c|}{1+|c|^2}\,[z,w]\,. \end{aligned} \tag{1.6}$$

On the other hand[4],

$$\begin{aligned} \frac{1+|c|^2}{|c|}\,[z,w] &= \frac{|c||z-w|}{\frac{|c|^2}{1+|c|^2}\sqrt{(1+|z|^2)(1+|w|^2)}} \\ &= \frac{|cz-cw|}{\sqrt{\left(\frac{|c|^2}{1+|c|^2} + \frac{|cz|^2}{1+|c|^2}\right)\left(\frac{|c|^2}{1+|c|^2} + \frac{|cw|^2}{1+|c|^2}\right)}}. \end{aligned} \tag{1.7}$$

Since, $z, w, c \in \mathbb{C}$, we deduce that

$$\frac{|c|^2}{1+|c|^2} < 1, \quad \frac{|cz|^2}{1+|c|^2} \le |cz|^2, \quad \frac{|cw|^2}{1+|c|^2} \le |cw|^2.$$

Hence, we see from (1.7) that

$$\frac{1+|c|^2}{|c|}[z,w] \ge [cz, cw]. \tag{1.8}$$

Method 2. Inequality (1.6) implies (1.8) by a change of variables:

Replace in (1.6) the numbers z, w and $1/c$ by z/c, w/c and d, respectively, to obtain

$$[z,w] \ge \frac{|c|}{1+|c|^2}\,[z/c, w/c] = \frac{1/|c|}{1/|c|^2+1}\,[z/c, w/c] = \frac{|d|}{1+|d|^2}\,[dz, dw]\,.$$

This is inequality (1.8).

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[4]Thanks to Hui Yu and Hongqiang Tu for this solution!

Solution 1.3. First, we show that there can be exceptional points.

Example 1. Let

$$a_n(z) = \frac{2^n(z-2)+\pi}{2^n(z-2)+1} \cdot \frac{2^n(z-3)+e}{2^n(z-3)+1}.$$

Then

$$a_n(z) \to \begin{cases} 1, & z \in \mathbb{C} \setminus \{2,3\}, \\ \pi, & z = 2, \\ e, & z = 3, \end{cases}$$

as $n \to \infty$.

Example 2. Let d be a positive integer and let $b_1, \ldots, b_d \in \mathbb{C}$. Let

$$a_n(z) = \prod_{k=1}^{d} \frac{2^n(z-k)+b_k}{2^n(z-k)+1}.$$

Then a_n is a rational function of degree d and

$$a_n(z) \to \begin{cases} 1, & z \in \mathbb{C} \setminus \{1, \ldots, d\}, \\ b_k, & z = k \in \{1, \ldots, d\}, \end{cases}$$

as $n \to \infty$. Hence $a_n(z) \to a(z) \equiv 1$ locally uniformly outside a finite set of points.

Example 3. Let $d \in \mathbb{N}$, let $z_1, \ldots, z_d \in \mathbb{C}$ be distinct and let $b_1, \ldots, b_d \in \mathbb{C}$. Let a be some rational function of degree d. Set

$$a_n(z) = a(z) + \left(\prod_{k=1}^{d} \frac{2^n(z-z_k)+(b_k+1)}{2^n(z-z_k)+1} \right) - 1.$$

Then $a_n(z)$ is a rational function of degree d and

$$a_n(z) \to \begin{cases} a(z), & z \notin \{z_1, \ldots, z_q\}, \\ a(z_k)+b_k, & z = z_k \in \{z_1, \ldots, z_q\}, \end{cases}$$

as $n \to \infty$.

Method I. We summarize facts from Schiff - Normal families, pages 208-213.

Definition 1.7. Let $\mathcal{F}$ be a family of meromorphic functions on a domain $\Omega \subset \mathbb{C}$. Then $\mathcal{F}$ is *quasi-normal* on Ω if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges spherically uniformly on compact subsets of $\Omega \setminus E$, where E is a (possibly empty) finite set of points of Ω. This set E of exceptional points may vary with the particular sequence and constitutes the set of *irregular points*.

Due to the uniform convergence on compact subsets, the convergence is either to an meromorphic function or to the function $\equiv \infty$ on $\Omega \setminus E$. If the set of irregular points never surpasses q in number, yet for some sequence there are q such points, then q is the *order* of the quasi-normal family.

Theorem 1.8 (Theorem A.9 in Schiff - Normal families). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain Ω which take values a, b, c at most p, q, r times, respectively, with $p \geq q \geq r$. Then $\mathcal{F}$ is quasi-normal of order at most q.*

Solution of Exercise 1.3. Since $a_j \in \mathcal{R}_d$ attains values $1, 2, 3$ at most d times, by Theorem 1.8, $\{a_j\}_{j \in \mathbb{N}}$ is quasi-meromorphic of order d. Hence there is a subsequence and a set

E of d points such that a_j converge spherically uniformly to a meromorphic function a or to a constant function $\equiv \infty$ in $\mathbb{C} \setminus E$. By Hurwitz' theorem the limit function a satisfies $a \in \mathcal{R}_d$.

THEOREM 1.9 (Hurwitz). *Let $\{f_k\}$ be a sequence of holomorphic functions on a connected open set G that converge uniformly on compact subsets of G to a holomorphic function $f \not\equiv 0$. If f has a zero of order m at z_0 then for every small enough $\rho > 0$ and for sufficiently large $k = k(\rho) \in \mathbb{N}$, f_k has precisely m zeroes in the disk defined by $|z - z_0| < \rho$, including multiplicity. Furthermore, these zeroes converge to z_0 as $k \to \infty$.*

Theorem 1.9 can be proved by using Morera's theorem and argument principle.

Method II. (i) Assume that

$$B = \sup_{k \in \mathbb{N}} |b_k| < \infty, \quad C = \sup_{k \in \mathbb{N}} |c_k| < \infty.$$

Then for some subsequence b_m

$$b_m \to b \in \mathbb{C}, \quad m \to \infty.$$

From the obtained sequence $(c_m)_{m \in \mathbb{N}}$ we find another subsequence c_n such that

$$c_n \to c \in \mathbb{C}, \quad n \to \infty.$$

Hence

$$b_n \to b \in \mathbb{C}, \quad c_n \to c \in \mathbb{C}$$

as $n \to \infty$. Hence

$$b_n z + c_n \to bz + c.$$

If $b = 0$, then this is a constant function.

If $B = \infty$, $C < \infty$, then

$$b_n z + c_n \to \infty, \quad n \to \infty, \quad z \in \mathbb{C} \setminus \{0\}.$$

If $B < \infty$, $C = \infty$, then

$$b_n z + c_n \to \infty, \quad n \to \infty, \quad z \in \mathbb{C}.$$

If $B = C = \infty$, and

$$b_n z_0 + c_n \not\to \infty, \quad n \to \infty,$$

for some $z_0 \in \mathbb{C}$, then

$$b_n z + c_n = b_n(z - z_0) + (b_n) z_0 + c_n \to \infty, \quad n \to \infty, \quad z \neq z_0.$$

Hence, in the case $B = C = \infty$, the sequence $b_n z + c_n$ converges in the whole plane except possibly at one point. The exception z_0 can exist. For example, if $z = -2$, then

$$2^n z + 2^{n+1} = -2^{n+1} + 2^{n+1} = 0 \not\to \infty, \quad n \to \infty.$$

Therefore the assertion holds.

(ii) We see that outside a finite set E, for some subsequences,

$$b_j + z a_j$$

converges to a polynomial of degree 1 (can be constant),

$$z(b_j + z a_j)$$

converges to a polynomial of degree 2,

$$c_j + z(b_j + z a_j)$$

converges to a polynomial of degree 2,

$$z(c_j + z(b_j + za_j))$$

converges to a polynomial of degree 3, and

$$d_j + z(c_j + z(b_j + za_j))$$

converges to a polynomial of degree 3.

(iii) By the proof of (ii) we deduce that for some subsequences $p_n \to p$ and $q_n \to q$, where p and q are polynomials of degree d. Convergence happens outside a finite set E. Now, by the triangle inequality of the chordal metric and (1.5), we have

$$\begin{aligned}\left[\frac{p_n}{q_n}, \frac{p}{q}\right] &\le \left[\frac{p_n}{q_n}, \frac{p}{q_n}\right] + \left[\frac{p}{q_n}, \frac{p}{q}\right] = \left[\frac{p_n}{q_n}, \frac{p}{q_n}\right] + \left[\frac{q_n}{p}, \frac{q}{p}\right] \\ &\le \frac{|q_n|^2+1}{|q_n|}\left[p_n, p\right] + \frac{|p|^2+1}{|p|}\left[q_n, q\right].\end{aligned} \tag{1.9}$$

Hence

$$\left[\frac{p_n(z)}{q_n(z)}, \frac{p(z)}{q(z)}\right] \le \frac{|q_n(z)|^2+1}{|q_n(z)|}\left[p_n(z), p(z)\right] + \frac{|p(z)|^2+1}{|p(z)|}\left[q_n(z), q(z)\right] \to 0,$$

as $n \to \infty$, for all $z \in \mathbb{C} \setminus E$ such that $p(z), q(z) \neq 0, \infty$. The assertion follows.

The fact that the convergence is uniform is not totally clear from this proof.

However, the convergence seems to be locally uniform. Since the Riemann sphere is compact in the spherical metric (open cover of a compact subset has a finite subcover), in this case, the locally uniform convergence implies uniform convergence.

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Solution 1.4. (i) We calculate

$$\frac{d}{dt}\overline{\varphi(t)} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{\overline{\varphi(t+h)} - \overline{\varphi(t)}}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \overline{\frac{\varphi(t+h)-\varphi(t)}{h}} = \overline{\varphi'(t)}.$$

(ii) By (i), we have

$$\frac{d}{dt}\operatorname{Re}(\varphi(t)) = \frac{d}{dt}\frac{1}{2}\left(\varphi(t) + \overline{\varphi(t)}\right) = \frac{1}{2}\left(\varphi'(t) + \overline{\varphi'(t)}\right) = \operatorname{Re}(\varphi'(t)).$$

(iii) By (i), we have

$$\frac{d}{dt}|\varphi(t)|^2 = \frac{d}{dt}\varphi(t)\overline{\varphi(t)} = \varphi'(t)\overline{\varphi(t)} + \varphi(t)\overline{\varphi'(t)} = 2\operatorname{Re}(\varphi'(t)\overline{\varphi(t)}.$$

Alternative solution. Let $\varphi(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real analytic functions in t.

(i) We have

$$\frac{d}{dt}\overline{\varphi(t)} = \frac{d}{dt}(x(t) - iy(t)) = x'(t) - iy'(t)$$

and

$$\overline{\varphi'(t)} = \overline{x'(t) + iy'(t)} = x'(t) - iy'(t).$$

Therefore, $\frac{d}{dt}\overline{\varphi(t)} = \varphi'(t)$.

(ii) We have

$$\frac{d}{dt}\operatorname{Re}(\varphi(t)) = \frac{d}{dt}x(t) = x'(t)$$

and

$$\operatorname{Re}(\varphi'(t)) = \operatorname{Re}(x'(t) + iy'(t)) = x'(t).$$

Therefore, $\frac{d}{dt}\operatorname{Re}(\varphi(t)) = \operatorname{Re}(\varphi'(t))$.

(iii) We have

$$\frac{d}{dt}|\varphi(t)|^2 = \frac{d}{dt}(x^2(t) + y^2(t)) = 2(x(t)x'(t) + y(t)y'(t))$$

and

$$2\operatorname{Re}(\varphi'(t)\overline{\varphi'(t)}) = 2\operatorname{Re}\left[(x'(t) + iy'(t))(x(t) - iy(t))\right] = 2(x(t)x'(t) + y(t)y'(t))$$

Therefore, $\frac{d}{dt}|\varphi(t)| = 2\operatorname{Re}(\varphi'(t)\overline{\varphi(t)})$.

* * *

Solution 1.5. We first check the equality

$$P(z,\theta) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \operatorname{Re}\left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right)$$

We see that

$$\begin{aligned}\frac{1+z}{1-z} &= \frac{(1+z)\overline{(1-z)}}{|1-z|^2} = \frac{(1+z)(1-\overline{z})}{|1-z|^2} = \frac{1 - \overline{z} + z - |z|^2}{|1-z|^2} \\ &= \frac{1-|z|^2}{|1-z|^2} + \frac{z - \overline{z}}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} + i\frac{2\operatorname{Im}(z)}{|1-z|^2},\end{aligned}$$

where for $z = x + iy$, $z - \overline{z} = 2iy = 2i\operatorname{Im}(z)$. Hence

$$\operatorname{Re}\frac{1+w}{1-w} = \frac{1-|w|^2}{|1-w|^2}, \quad w \in \mathbb{C}\setminus\{1\}.$$

Set $w = ze^{-i\theta}$ to obtain

$$\operatorname{Re}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right) = \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = P(z,\theta).$$

(i) Since

$$P(te^{i\theta}, 0) = \frac{1 - |te^{i\theta}|^2}{|1 - te^{i\theta}|^2} = \frac{1-t^2}{|1-te^{i\theta}|^2},$$

where

$$|1 - te^{i\theta}|^2 = |1 - t\cos\theta + it\sin\theta|^2 = (1 - t\cos\theta)^2 + t^2\sin\theta = 1 - 2t\cos\theta + t^2,$$

we obtain

$$\frac{\partial}{\partial\theta}P(te^{i\theta},0) = \frac{-(1-t^2)\frac{d}{d\theta}|1-te^{i\theta}|^2}{|1-te^{i\theta}|^4} = \frac{-(1-t^2)\cdot 2t\sin\theta}{|1-te^{i\theta}|^4} = -\frac{2t(1+t)(1-t)\sin\theta}{|1-te^{i\theta}|^4}.$$

(ii) Let $0 < a < 1$. We have

$$g(te^{i\theta}, a) = \frac{1}{2}\log\left|\frac{1 - ate^{i\theta}}{a - te^{i\theta}}\right|^2 = \frac{1}{2}(\log|1 - ate^{i\theta}|^2 - \log|1 - ate^{i\theta}|^2) = \frac{1}{2}(A(\theta) - B(\theta)).$$

Here

$$A(\theta) = \log(1 - 2at\cos\theta + a^2t^2)$$

and

$$B(\theta) = \log(a^2 - 2at\cos\theta + t^2).$$

Hence

$$A'(\theta) = \frac{2at\sin\theta}{|1 - ate^{i\theta}|^2}, \quad B'(\theta) = \frac{2at\sin\theta}{|a - te^{i\theta}|^2}.$$

Hence

$$\frac{\partial}{\partial\theta} g(te^{i\theta}, a) = \frac{at\sin\theta}{|1 - ate^{i\theta}|^2} - \frac{at\sin\theta}{|a - te^{i\theta}|^2}.$$

Some alternative solutions. (i) Denote

$$\varphi(\theta) = \frac{1}{1 - te^{i\theta}}, \quad \varphi'(\theta) = \frac{te^{i\theta}i}{(1 - te^{i\theta})^2}.$$

Since $\frac{d}{dt} \operatorname{Re} |f(t)|^2 = 2 \operatorname{Re} f'(t)\overline{f(t)}$ by Exercise 1.4(iii),

$$\frac{d}{d\theta}|\varphi(\theta)|^2 = 2\operatorname{Re}\varphi'(\theta)\varphi(\theta) = 2\operatorname{Re}\frac{te^{i\theta}i(1 - te^{-i\theta})}{|1 - te^{i\theta}|^4}.$$

Here

$$\begin{aligned} te^{i\theta}i(1 - te^{-i\theta}) &= te^{i\theta}i - t^2 i \\ &= ti\cos\theta - t\sin\theta - t^2 i \\ &= -t\sin\theta + i[t\cos\theta - t^2]. \end{aligned}$$

Hence

$$\frac{d}{d\theta}|\varphi(\theta)|^2 = \frac{-2t\sin\theta}{|1 - te^{i\theta}|^4}$$

and

$$\frac{d}{d\theta}P(te^{i\theta}, 0) = (1 - t^2)\frac{d}{d\theta}|\varphi(\theta)|^2 = \frac{-2t(1 + t)(1 - t)\sin\theta}{|1 - te^{i\theta}|^4}.$$

(i) We start from

$$P(te^{i\theta}, 0) = \operatorname{Re}\left(\frac{1 + te^{i\theta}}{1 - te^{i\theta}}\right).$$

Since $\frac{d}{dt} \operatorname{Re} f(t) = \operatorname{Re} f'(t)$ by Exercise 1.4(ii)

$$\begin{aligned} \frac{d}{d\theta}P(te^{i\theta}, 0) &= \operatorname{Re}\frac{(1 - te^{i\theta})te^{i\theta}i - (1 + te^{i\theta})(-te^{i\theta}i)}{(1 - te^{i\theta})^2} \\ &= \operatorname{Re}\frac{2te^{i\theta}i}{(1 - te^{i\theta})^2} = 2\operatorname{Re}\frac{te^{i\theta}i(1 - te^{-i\theta})^2}{|1 - te^{i\theta}|^4} \\ &= 2\operatorname{Re}\frac{te^{i\theta}i - 2t^2 i + t^3 e^{-i\theta}i}{|1 - te^{i\theta}|^4} \\ &= 2\operatorname{Re}\frac{-t\sin\theta + t^3\sin\theta}{|1 - te^{i\theta}|^4} \\ &= \frac{-2t(1 + t)(1 - t)\sin\theta}{|1 - te^{i\theta}|^4}. \end{aligned}$$

We used Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ and the fact that cos is even and sin is odd.

(ii) Let $0 < a < 1$. Since $\log z = \log|z| + i\arg z$, we have

$$g(z, a) = \log\frac{1}{|\varphi_a(z)|} = -\operatorname{Re}\log\varphi_a(z), \quad \varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Hence (by Exercise 1.4(ii))

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = -\operatorname{Re}\left(\frac{\varphi_a'(z)}{\varphi_a(z)} te^{i\theta} i\right)$$

Here

$$\begin{aligned}\frac{\varphi_a'(z)}{\varphi_a(z)} &= \frac{d}{dz}\log\varphi_a(z)\\ &= \frac{d}{dz}(\log(a-z) - \log(1-\bar{a}z))\\ &= \frac{-1}{a-z} + \frac{\bar{a}}{1-\bar{a}z}\\ &= \frac{\bar{a}(1-a\bar{z})}{|1-\bar{a}z|^2} - \frac{\bar{a}-\bar{z}}{|a-z|^2}.\end{aligned}$$

Since $0 < a < 1$ satisfies $\bar{a} = a$, we obtain

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = -\operatorname{Re}\left(\frac{\varphi_a'(z)}{\varphi_a(z)} te^{i\theta} i\right) = -\operatorname{Re} te^{i\theta} i\left(\frac{a(1-ate^{-i\theta})}{|1-ate^{i\theta}|^2} - \frac{a-te^{-i\theta}}{|a-te^{i\theta}|^2}\right)$$

Here

$$\begin{aligned}-te^{i\theta} ia(1-ate^{-i\theta}) &= -a\left[te^{i\theta} i - at^2 i\right]\\ &= -a\left[it\cos\theta - t\sin\theta - at^2 i\right]\\ &= \underline{at\sin\theta} + i\left[a^2t^2 - at\cos\theta\right]\end{aligned}$$

and

$$\begin{aligned}-te^{i\theta} i(-(a-te^{-i\theta})) &= te^{i\theta} i(a-te^{-i\theta})\\ &= ate^{i\theta} i - t^2 i\\ &= ati\cos\theta - at\sin\theta - t^2 i\\ &= \underline{-at\sin\theta} + i\left[at\cos\theta - t^2\right].\end{aligned}$$

Hence

$$\frac{\partial g}{\partial \theta}(te^{i\theta}, a) = \frac{at\sin\theta}{|1-ate^{i\theta}|^2} - \frac{at\sin\theta}{|a-te^{i\theta}|^2}.$$

Remark. Many times the trigonometric estimates are difficult. Then, it is useful to estimate

$$|1-re^{it}|^p \asymp (1-r)^p + |t|^p.$$

Here $A \asymp B$ if $C^{-1}A \le B \le CA$ for some $C \in (0, \infty)$.