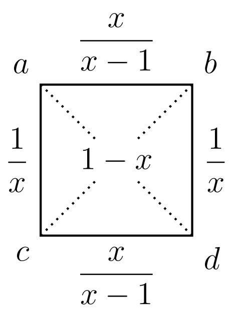
Research Seminar in Mathematics 2019
Exercise session: 26.4. at 12–14 in room M105

Exercise 2.1 The *cross-ratio* is defined as

$$\operatorname{cr}(a,b,c,d) = \frac{(a-b)(c-d)}{(a-d)(c-b)}, \quad a,b,c,d \in \overline{\mathbb{C}}.$$

Denote $x = \operatorname{cr}(a,b,c,d)$. Show that



(a) $\operatorname{cr}(c,b,a,d) = \dfrac{1}{x}$,

(b) $\operatorname{cr}(d,b,c,a) = 1-x$,

(c) $\operatorname{cr}(b,a,c,d) = \dfrac{x}{x-1}$,

(d) $\operatorname{cr}(a,b,c,d) = \operatorname{cr}(c,d,a,b)$.

By (a)–(d), it follows that permuting arguments changes the value according to the enclosed diagram. Solution.

Exercise 2.2 Let $x = \operatorname{cr}(a,b,c,d)$. What is the value of

(a) $\operatorname{cr}(c,a,b,d)$?

(b) $\operatorname{cr}(b,c,a,d)$? Solution.

Exercise 2.3 Let $a,b,c,d,x \in \mathbb{C}$ be distinct. Show[1] that

(a) $\dfrac{\operatorname{cr}(a,b,c,d)}{\operatorname{cr}(x,b,c,d)} = \operatorname{cr}(a,b,x,d)$,

(b) $\operatorname{cr}(a,b,c,d) - \operatorname{cr}(x,b,c,d) = \dfrac{\operatorname{cr}(a,b,c,d)-1}{\operatorname{cr}(a,c,d,x)}$. Solution.

Exercise 2.4 The curvature of the metric $\rho(z)|dz|$ at a point z is given by

$$K(z) = -\frac{(\Delta \log \rho(z))(z)}{\rho^2(z)}, \quad \text{where } \Delta = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

is the Laplacian in the polar coordinates. Show that if

$$\rho(z) = \rho_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}, \tag{2.1}$$

then the curvature $K(z) \equiv -4$. Solution.

Exercise 2.5 For distinct $a_1, \ldots, a_q \in \mathcal{R}_d$, where $d \geq 1$ and $q \geq 3$, we set

$$X(a_1, \ldots, a_q) = \{z \in \mathbb{C} \setminus \{0,1\} \; ; \; a_i(z) \neq a_j(z) \text{ for } i \neq j\}.$$

Thus, X is the Riemann sphere with p punctures.

(a) Show that $3 \leq p \leq 2d \times \dfrac{q(q-1)}{2} + 3$.

[1] Identities (a) and (b) are needed in Yamanoi's paper on pages 50 and 31, respectively.

(b) Show[2] that $A_{\mathrm{hyp}}(X(a_1,\dots,a_q)) = \frac{\pi}{2}(p-2) \le 2dq^2$. Solution.

Exercise 2.6 For the unit disc $\mathbb{D} = \{|z|<1\}$, we define the hyperbolic density as

$$\rho(z) = \rho_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}.$$

For other domains, the hyperbolic density can be calculated with the following formula. If $f: A \to B$ is conformal and $g = f^{-1}$, then we obtain for the hyperbolic densities

$$\rho_B(z) = \rho_A(g(z))|g'(z)|.$$

For example, the map $f: \mathbb{D} \to D(0,2)$, $f(z) = 2z$ is a conformal bijection. Its inverse is $g(z) = z/2$. Now $g'(z) = 1/2$. We obtain

$$\rho_{D(0,2)}(z) = \rho_{\mathbb{D}}(g(z))|g'(z)| = \frac{1}{1-\left|\frac{z}{2}\right|^2} \cdot \frac{1}{2} = \frac{2}{2^2-|z|^2}.$$

(a) By using

$$a: \mathbb{D} \to \mathbb{H} = \{0 < \mathrm{Im}(z)\}, \quad a(z) = i\frac{1+z}{1-z},$$

show that

$$\rho_{\mathbb{H}}(z) = \frac{1}{2\,\mathrm{Im}(z)}.$$

(b) By using

$$b: \mathbb{H} \to L = \left\{z \in \mathbb{C}\,;\, -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\right\}, \quad b(z) = \log(-iz),$$

show that

$$\rho_L(z) = \frac{1}{2\cos \mathrm{Im}(z)}.$$

(c) By using

$$c: L \to S = \{z \in \mathbb{C}\,;\, 0 < \mathrm{Im}(z) < \lambda\}, \quad c(z) = \left(z + i\frac{\pi}{2}\right)\frac{\lambda}{\pi},$$

show that

$$\rho_S(z) = \frac{\pi}{\lambda \sin\left(\frac{\pi}{\lambda}\mathrm{Im}(z)\right)}.$$

(d) By using

$$d: S \to A(0,1,e^{\lambda}) = \left\{z \in \mathbb{C}\,;\, 1 < |z| < e^{\lambda}\right\}, \quad d(z) = e^{iz},$$

show that

$$\rho_{A(0,1,e^{\lambda})}(z) = \frac{\pi}{2\lambda|z|\sin\left(\frac{\pi}{\lambda}\log|z|\right)}.$$

In this case, $d: S \to A(0,1,e^{\lambda})$ is not conformal, that is, an analytic bijection. However, d is a so-called universal covering map. Also, in this case, it is possible to find a function g such that $d(g(z)) = z$ and proceed. Solution.

[2]Gauss-Bonnet theorem states that

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M).$$

In our case, $M = X(a_1,\dots,a_q)$, $K \equiv -4$ and ∂M has measure zero.

Exercise 2.7 (a) Show that as $\lambda \to \infty$,

$$\rho_S(z) = \frac{\pi}{\lambda \sin\left(\frac{\pi}{\lambda}\operatorname{Im}(z)\right)} \to \frac{1}{2\operatorname{Im}(z)} = \rho_{\mathbb{H}}(z)$$

(b) Let $R > 1$. Show that as $R \to \infty$,

$$\rho_{A(0,1,R)} = \frac{\pi/\log R}{2\lambda \sin\left(\pi \log|z|/\log R\right)} \to \frac{1}{2|z|\log|z|} = \rho_{A(0,1,\infty)}(z)$$

(c) By using the map,

$$f : A(0,1,\infty) \to A(0,0,1), \quad f(z) = \frac{1}{z},$$

show that

$$\rho_{A(0,0,1)} = \frac{1}{2|z|\log\frac{1}{|z|}}.$$

Solution.

For the various domains in Exercises 2.6 and 2.7, see Figure 1 on the next page.

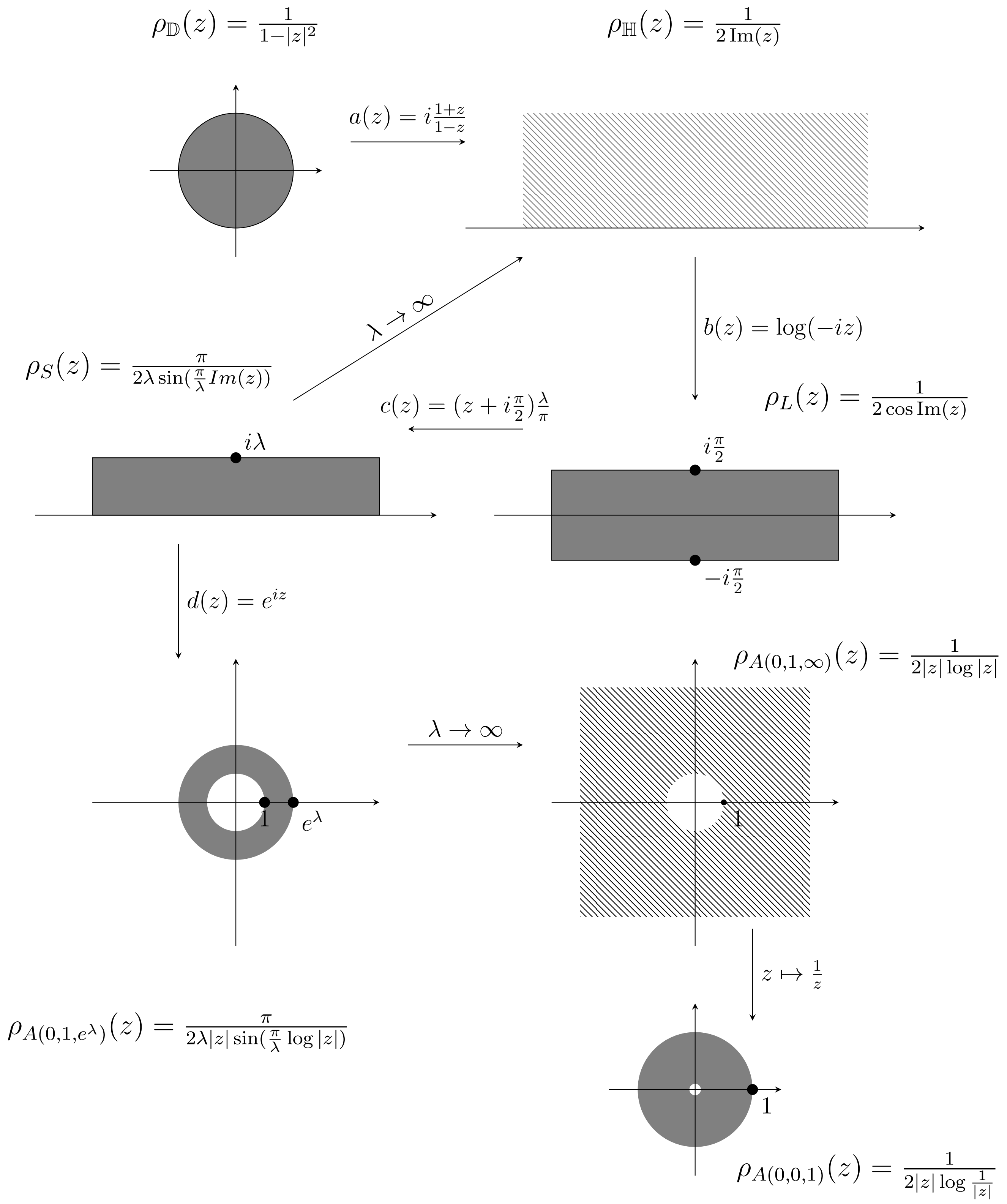


FIGURE 2.1. Hyperbolic densities of various domains.

Solutions

Solution 2.1. (a) We see that

$$\operatorname{cr}(c,b,a,d) = \frac{(c-b)(a-d)}{(c-d)(a-b)} = \left(\frac{(a-b)(c-d)}{(a-d)(c-b)}\right)^{-1} = \frac{1}{\operatorname{cr}(a,b,c,d)}.$$

(b) We calculate

$$\begin{aligned} 1-\operatorname{cr}(a,b,c,d) &= 1-\frac{(a-b)(c-d)}{(a-d)(c-b)} \\ &= \frac{(\cancel{ac}-ab-dc+\cancel{db})-(\cancel{ac}-ad-bc+\cancel{bd})}{(a-d)(c-b)} \\ &= \frac{(a-c)(d-b)}{(a-d)(b-c)} = \operatorname{cr}(a,c,b,d) \\ &= \frac{(d-b)(c-a)}{(d-a)(c-b)} = \operatorname{cr}(d,b,c,a). \end{aligned}$$

(c) Let $\operatorname{cr}(a,b,c,d)=x$. By (a), we have

$$\operatorname{cr}(c,b,a,d) = \frac{1}{x}.$$

Now, (b) yields

$$\operatorname{cr}(c,a,b,d) = 1-\left(\frac{1}{x}\right) = \frac{x-1}{x}.$$

Now, (a) yields

$$\operatorname{cr}(b,a,c,d) = \left(\frac{x-1}{x}\right)^{-1} = \frac{x}{x-1}.$$

(d) We see that

$$\operatorname{cr}(c,d,a,b) = \frac{(c-d)(a-b)}{(c-b)(a-d)} = \frac{(a-b)(c-d)}{(a-d)(c-b)} = \operatorname{cr}(a,b,c,d).$$

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Solution 2.2. (a) Let $y=\operatorname{cr}(c,a,b,c)$. Then by the diagram, we have

$$\operatorname{cr}(a,c,b,d) = \frac{y}{y-1}.$$

Then by the diagram, we have

$$\operatorname{cr}(a,b,c,d) = 1-\frac{y}{y-1} = \frac{y-1-y}{y-1} = \frac{1}{1-y} = x.$$

We solve

$$\operatorname{cr}(c,a,b,c) = y = \frac{x-1}{x}.$$

(b) Let $y=\operatorname{cr}(b,c,a,d)$. Then by the diagram, we have

$$\operatorname{cr}(a,c,b,d) = \frac{1}{y}.$$

Then by the diagram, we have

$$\operatorname{cr}(a,b,c,d) = 1-\frac{1}{y} = x.$$

We solve

$$\operatorname{cr}(b,c,a,d) = y = \frac{1}{1-x}.$$

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Solution 2.3. (a) Let $\alpha = x$. We calculate

$$\begin{aligned}\frac{\operatorname{cr}(a,b,c,d)}{\operatorname{cr}(\alpha,b,c,d)} &= \frac{(a-b)(c-d)}{(a-d)(c-b)} : \frac{(\alpha-b)(c-d)}{(\alpha-d)(c-b)} \\ &= \frac{(a-b)(\alpha-d)}{(a-d)(\alpha-b)} = \operatorname{cr}(a,b,\alpha,d).\end{aligned}$$

(b) We calculate

$$\begin{aligned}&\operatorname{cr}(a,b,c,d) - \operatorname{cr}(\alpha,b,c,d) \\ &= \frac{(a-b)(c-d)}{(a-d)(c-b)} - \frac{(\alpha-b)(c-d)}{(\alpha-d)(c-b)} \\ &= \frac{c-d}{c-b} \times \frac{(a-b)(\alpha-d)-(a-d)(\alpha-b)}{(a-d)(\alpha-d)} \\ &= \frac{c-d}{c-b} \times \frac{(\cancel{a\alpha} - ad - b\alpha - \cancel{bd}) - (\cancel{a\alpha} - ab - d\alpha - \cancel{db})}{(a-d)(\alpha-d)} \qquad (2.2) \\ &= \frac{c-d}{c-b} \times \frac{(a-\alpha)(b-d)}{(a-d)(\alpha-d)} \\ &= \frac{(b-d)(c-d)}{(a-d)(c-b)} \times \frac{a-\alpha}{\alpha-d}\end{aligned}$$

and note

$$\operatorname{cr}(a,b,c,d) - 1 = -\operatorname{cr}(a,c,b,d) = -\frac{(a-c)(b-d)}{(a-d)(b-c)} = \frac{(a-c)(b-d)}{(a-d)(c-b)} \qquad (2.3)$$

By dividing (2.3) with (2.2), we obtain

$$\begin{aligned}\frac{\operatorname{cr}(a,b,c,d)-1}{\operatorname{cr}(a,b,c,d)-\operatorname{cr}(\alpha,b,c,d)} &= \frac{(a-c)(\alpha-d)}{(c-d)(a-\alpha)} = \frac{(a-c)(d-\alpha)}{(a-\alpha)(d-c)} \\ &= \operatorname{cr}(a,c,d,\alpha).\end{aligned}$$

This can be rewritten as

$$\operatorname{cr}(a,b,c,d) - \operatorname{cr}(\alpha,b,c,d) = \frac{\operatorname{cr}(a,b,c,d)-1}{\operatorname{cr}(a,c,d,\alpha)}.$$

* * *

Solution 2.4. We recall that the Laplacian in polar coordinates is

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}. \qquad (2.4)$$

Hence, for

$$\rho(z) = \frac{1}{1-|z|^2} = \frac{1}{1-r^2},$$

we obtain

$$\log\rho(r) = -\log(1-r^2), \quad (\log\rho)_r = \frac{2r}{1-r^2}, \quad (\log\rho)_{rr} = \frac{2}{1-r^2} + \frac{4r^2}{(1-r^2)^2}.$$

By (2.4), we obtain

$$\Delta(\log\rho) = \left(\frac{2}{1-r^2} + \frac{4r^2}{(1-r^2)^2}\right) + \frac{2}{1-r^2} = \frac{4}{(1-r^2)^2}$$

and consequently $K(z) = -4$.

If [3] the value distances would be scaled up by a factor of 2, K would have the value -1.

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Solution 2.5. For distinct $a_1, \ldots, a_q \in \mathcal{R}_d$, where $d \geq 1$ and $q \geq 3$, we set

$$X(a_1, \ldots, a_q) = \{z \in \mathbb{C} \setminus \{0,1\} \ ; \ a_i(z) \neq a_j(z) \text{ for } i \neq j\}.$$

Thus, X is the Riemann sphere with p punctures.

(a) **Claim:** $3 \leq p \leq 2d \times \dfrac{q(q-1)}{2} + 3$.

First, we see that, in every case, three points $0, 1, \infty \notin X(a_1, \ldots, a_q)$. Hence $p \geq 3$ is clear.

Second, let $a_k = p_k/q_k$ for $k = 1, \ldots, q$. The condition

$$a_i(z) = \frac{p_i(z)}{q_i(z)} = \frac{p_j(z)}{q_j(z)} = a_j(z)$$

is equivalent to

$$p_i(z)q_j(z) - p_j(z)q_i(z) = 0,$$

which has at most $2d$ solutions, since the function on the left is a polynomial of degree at most $2d$. The number of pairs (a_i, a_j), $i \neq j$, is

$$\binom{q}{2} = \frac{q!}{2!(q-2)!} = \frac{q(q-1)}{2}.$$

Therefore

$$a_i(z) = a_j(z), \quad i \neq j,$$

has at most

$$2d \cdot \frac{q(q-1)}{2}$$

solutions. No such solution z needs to be $0, 1, \infty$, so there are *three* more points. Therefore, in conclusion,

$$p \leq 2d \times \frac{q(q-1)}{2} + 3.$$

(b) This exercise was quite difficult, since the Gauss-Bonnet theorem is not very familiar. We will go through the background in the formula, take a few examples of its use and then solve our exercise.

[3] It can be shown (see Hubbard's book about Teichmüller spaces) that $K(x)$ at a point x is a number such that

$$\text{Area}(D_r) = \pi\left(r^2 - \frac{1}{12}K(x)r^4\right) + o(r^4).$$

This shows that if $r = 2s$, then $K_s = 4K_r$.

Background. The Gauss-Bonnet theorem is

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M),$$

where

- M is a surface;
- ∂M is its boundary;
- K is the Gaussian curvature of M;
- k_g is the geodesic curvature of ∂M;
- $\chi(M)$ is the Euler characteristic of M (usually $\chi(M) = 2(1 - \#\text{holes})$.

Curvature in plane. If a plane curve is given as $y = g(x)$, then its curvature is

$$\kappa = \frac{g''}{(1 + (g')^2)^{3/2}}.$$

Example. We study the upper half of

$$x^2 + y^2 = r^2.$$

In this case,

$$g(x) = -\sqrt{r^2 - x^2}, \quad g'(x) = \frac{x}{\sqrt{r^2 - x^2}}, \quad g''(x) = \frac{r^2}{(r^2 - x^2)^{3/2}}.$$

Hence

$$\kappa(x) = \frac{r^2}{(r^2 - x^2)^{3/2}} \Bigg/ \left(\frac{r^2}{r^2 - x^2}\right)^{3/2} = \frac{1}{r}.$$

We see that if a circle has small radius r, then its curvature $|\kappa|$ will be large.

Example. Let $a > 0$ and

$$y = g(x) = \frac{x^2}{a}.$$

Then the curve is a parabola going through points $(\pm a, a)$ and $(0, 0)$. We have

$$g'(x) = \frac{2x}{a}, \quad g''(x) = \frac{2}{a},$$

and consquently

$$\kappa(x) = \frac{2}{a} \Bigg/ \left(1 + \frac{4x^2}{a^2}\right)^{3/2}.$$

As a special case

$$\kappa(0) = \frac{2}{a},$$

which is the same value as of a circle with $r = \frac{a}{2}$. We see that $C(a/2, a/2)$ will touch the points $(0, a)$ and $(0, 0)$. We draw Figure 2.2.

Curvature of a surface. If we have a smooth surface $z = g(x, y)$, then for a point (x_0, y_0) on the surface, we can find crosscuts such that on one crosscut the curvature is maximal and on the other minimal. That is,

$$z = g(x, \alpha(x)), \quad z = g(x, \beta(x)),$$

where α, β are linear functions, such that $(x_0, y_0) = (x_0, \alpha(x_0)) = (x_0, \beta(x_0))$. Now for the plane curve

$$z = g(x, \alpha(x))$$

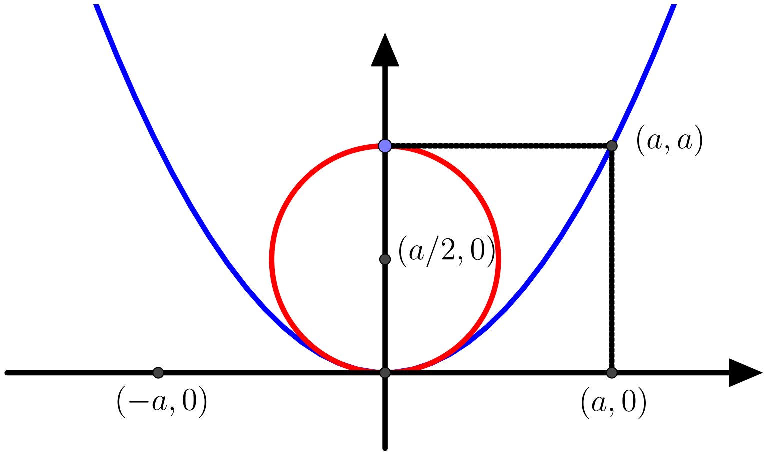

FIGURE 2.2. Parabola and an osculating /"kissing" circle.

we have $\kappa(x_0) = \kappa_{\max}$. Similarly for the other crosscut, $\kappa(x_0) = \kappa_{\min}$.

Definition. The Gaussian curvature $K(x_0, y_0) = \kappa_{\max} \cdot \kappa_{\min}$.

This is the same curvature as given by the formula

$$K = -\frac{(\Delta \log \rho)}{\rho^2},$$

but unfortunately, we cannot prove this here.

Example. Sphere of radius $r > 0$ has on the lower half

$$\kappa_{\max} = \kappa_{\min} = \frac{1}{r}$$

and therefore $K = \kappa_{\max}\kappa_{\min} = \frac{1}{r^2}$.

On the upper half,

$$K = \kappa_{\max}\kappa_{\min} = \left(\frac{-1}{r}\right)^2 = \frac{1}{r^2}.$$

Hence, the Gaussian curvature of a sphere is $1/r^2$.

Example.(a) The surface

$$z = \frac{x^2}{a} + \frac{y^2}{b}, \quad 0 < a < b,$$

satisfies

$$K(0) = \kappa_{\max}(0)\kappa_{\min}(0) = \frac{2}{a}\frac{2}{b} = \frac{4}{ab}.$$

Locally the surface is "similar" to a sphere.

(b) The surface

$$z = \frac{x^2}{a} - \frac{y^2}{b}, \quad 0 < a < b,$$

satisfies

$$K(0) = \kappa_{\max}(0)\kappa_{\min}(0) = \frac{2}{a} \cdot \frac{-2}{b} = -\frac{4}{ab}.$$

Locally the surface is a saddle surface.

Example.

$$\rho(z) = \frac{1}{1-|z|^2} \quad \to \quad K = -4$$

and

$$\rho(z) = \frac{1}{1+|z|^2} \quad \to \quad K = 4.$$

Geodesic curvature. In plane

$$k_g = \kappa = \frac{g''}{(1+(g')^2)^{3/2}}.$$

On some other surface, geodesic is the "locally shortest path between two points".

Example. The shortest path (around 300km) between Joensuu and Helsinki by plane is along a circle, whose center is in the center of Earth. So this path is a geodesic on Earth.

A person traveling from Joensuu to Helsinki can also "choose poorly" to travel through North pole, over Pacific ocean etc. and travel a long path (around 40000km-300km).

Example. Consider a circle of radius r in the plane, that is, $C(0,r) = \partial M$. Then

$$\int_{\partial M} k_g\, ds = k_g \int_{\partial M} ds = k_g \text{length}(\partial M) = \frac{1}{r} \cdot 2\pi r = 2\pi,$$

and we see that this integral is independent of the radius of the circle.

Example. Let $\gamma \subset \mathbb{R}^2$ be a path

$$\{x = r, -3 \le y \le 0\} \cup \left\{(r\cos\theta, r\sin\theta)\,;\, 0 \le \theta \le \frac{\pi}{2}\right\} \cup \{y = r, -3 \le x \le 0\}.$$

Then on the straight parts of the path γ, $k_g = 0$. On the other hand, on the circular part, $k_g = \frac{1}{r}$. The path γ is smooth. We have

$$\int_{\partial M} k_g\, ds = \frac{\pi}{2}.$$

By similar reasoning, as a limiting case, when $r \to 0$, we obtain that if two lines L_1, L_2 meet in an angle α, then

$$\int_{L_1 \cup L_2} k_g\, ds = \alpha.$$

Euler characteristic. A cube has

- vertices: $V = 8$;
- edges: $E = 12$;
- faces: $F = 6$.

By definition,

$$\chi(M) = V - E + F = 8 - 12 + 6 = 2.$$

A tetrahedron has

- vertices: $V = 4$;
- edges: $E = 6$;
- faces: $F = 4$.

Hence

$$\chi(M) = V - E + F = 4 - 6 + 4 = 2.$$

It can be shown that all solids "with no holes" have Euler characteristic $\chi(M) = 2$.

Example. The doughnut

$$([0,3] \times [0,3] \times [0,1]) \setminus ([1,2] \times [1,2] \times [0,1])$$

Has $V = 8 + 8 = 16$ and $F = 6 + 4 = 10$. We require that the faces are simply connected and hence make the cuts $(0,0,0) - -(1,1,0)$ and $(0,0,1) - -(1,1,1)$. After this, we have $E = 12 + 12 + 2 = 26$. Hence

$$\chi(M) = 16 - 26 + 10 = 0.$$

We see that for solids

$$\chi(M) = 2(1 - \#\text{holes}).$$

Since a surface can be triangulated, this allows a definition for $\chi(M)$ for general surface M.

If we have a simply connected plane region, then $\chi(M) = 1$, which is seen by considering a triangle:

$$\chi(\text{triangle}) = V - E + F = 3 - 3 + 1 = 1.$$

If we have a surface with punctures, then two punctures form one hole. We have

$$\chi(M) = 2(1 - \#\text{holes}) = 2 - \#\text{punctures}.$$

Gauss-Bonnet theorem. We will now verify Gauss-Bonnet theorem in a few examples.

Example. Consider a sphere M of radius r. Then $\partial M = \emptyset$. We have

$$\int_M K + \int_{\partial M} = \frac{1}{r^2} \cdot \text{area}(M) + 0 = \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi = 2\pi\chi(M),$$

since $\chi(M) = 2$. The Gauss-Bonnet theorem is seen to be valid.

Example. Consider a half sphere M of radius r. Then ∂M is a circle, which is a geodesic on the sphere. Hence $k_g \equiv 0$. We have

$$\int_M K + \int_{\partial M} = \frac{1}{r^2} \cdot \text{area}(M) + 0 = \frac{1}{r^2} \cdot \frac{4\pi r^2}{2} = 2\pi = 2\pi\chi(M),$$

since now $\chi(M) = 1$. The Gauss-Bonnet theorem is seen to be valid.

Example. Let M be a 1/8-part of a of radius r. Then ∂M consists of 3 geodesics which meet in angles $\pi/2$. Therefore

$$\int_M K + \int_{\partial M} = \frac{1}{r^2} \cdot \text{area}(M) + 3\frac{\pi}{2} = \frac{1}{r^2} \cdot \frac{4\pi r^2}{8} + \frac{3\pi}{2} = 2\pi = 2\pi\chi(M),$$

since $\chi(M) = 1$. The Gauss-Bonnet theorem is seen to be valid.

Solution of the exercise.

Now our surface is $M = X = \overline{\mathbb{C}} \setminus \{z_1, \ldots, z_p\}$, the Riemann sphere with p punctures. Now ∂X consists of discrete finite set of points and has measure zero. We have

$$\int_X K\, dA0 \int_{\partial X} k_g\, ds = -4\text{hyp-area}(X) + 0 = 2\pi\chi(X),$$

which yields
$$\text{hyp-area}(X) = \frac{2\pi}{-4}\chi(X) = \frac{\pi}{2}(-\chi(X)).$$
Now, $\chi(X) = 2 - \#\text{punctures} = 2 - p$ and therefore
$$\text{hyp-area}(X) = \frac{\pi}{2}(p-2).$$

Since
$$p \le 2d\frac{q(q-1)}{2} + 3,$$
we have
$$p - 2 = 2d\frac{q^2}{2} - 2d\left(\frac{-1}{2}\right) + 1 \le dq^2.$$
Hence
$$\text{hyp-area}(X) = \frac{\pi}{4}2(p-2) \le 1 \cdot 2dq^2 = 2dq^2.$$

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Solution 2.6. (a) The map $a : \mathbb{D} \to \mathbb{H}$,
$$a(z) = i\frac{1+z}{1-z}$$
is a bijection. Namely, 0 and ∞ are mirror images with respect to $\partial\mathbb{D}$. Their images $a(0) = i$ and $a(\infty) = -i$ are mirror images with respect to $\mathbb{R}$. Hence, $a(\partial\mathbb{D}) = \mathbb{R}$ and $a(0) = i$, so we must have $a(\mathbb{D}) = \mathbb{H}$.

Therefore
$$g(z) = a^{-1}(z) = \frac{z-i}{z+i}, \quad g'(z) = \frac{2i}{(z+i)^2}$$
We have
$$\rho_{\mathbb{H}}(z) = \rho_{\mathbb{D}}(g(z))|g'(z)| = \frac{\frac{2}{|z+i|^2}}{1 - \left|\frac{z-i}{z+i}\right|} = \frac{2}{|z+i|^2 - |z-i|^2}.$$
Here
$$|z+i|^2 - |z-i|^2 = 4\,\mathrm{Re}(\bar{i}z) = 4\,\mathrm{Im}(z).$$
Hence
$$\rho_{\mathbb{H}}(z) = \frac{1}{2\,\mathrm{Im}(z)}.$$

(b) Let
$$L = \left\{ z \in \mathbb{C}\,;\, -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2} \right\}.$$
Then $b : \mathbb{H} \to L$, $b(z) = \log(-iz)$ is conformal. Hence
$$g(z) = b^{-1}(z) = ie^z, \quad g'(z) = ie^z.$$
Now
$$\rho_L(z) = \rho_{\mathbb{H}}(g(z))|g'(z)| = \frac{|ie^z|}{2\,\mathrm{Im}(ie^z)}.$$
If $z = x + iy$, then
$$e^z = e^x e^i y = e^x(\cos(y) + i\sin(y)),$$

and therefore

$$|ie^z| = e^x, \quad \mathrm{Im}(ie^z) = e^x \cos(y),$$

so that

$$\rho_L(z) = \frac{1}{2\cos \mathrm{Im}(z)}.$$

We note that

$$\rho_{\mathbb{D}}(z) = \frac{1}{1-|z|^2} \to \infty, \quad z \to 1^-,$$

and $\cos(\pi/2) = \cos(-\pi/2) = 0$. Hence, the hyperbolic density is unbounded as the point approaches the boundary of the domain.

(c) Let

$$S = \{z \in \mathbb{C}\,;\, 0 < \mathrm{Im}(z) < \lambda\}.$$

Then $c : L \to S$,

$$c(z) = (z + i\frac{\pi}{2}) \cdot \frac{\lambda}{\pi}$$

is conformal. Hence

$$g(z) = z\frac{\pi}{\lambda} - i\frac{\pi}{2}, \quad g'(z) = \frac{\pi}{\lambda}.$$

We see that

$$\mathrm{Im}(g(z)) = \mathrm{Im}\left(z\frac{\pi}{\lambda} - i\frac{\pi}{2}\right)$$

and

$$\cos(x - \pi/2) = \cos(x)\cos(-\pi/2) - \sin(x)\sin(-\pi/2) = \sin x.$$

Therefore

$$\rho_S(z) = \rho_L(g(z))|g'(z)| = \frac{\pi}{2\lambda \cos\left(\frac{\pi}{\lambda}\mathrm{Im}(z) - \frac{\pi}{2}\right)} = \frac{\pi}{2\lambda \sin(\pi\,\mathrm{Im}(z)/\lambda)}.$$

(d) Let $S = \{z \in \mathbb{C}\,;\, 0 < \mathrm{Im}(z) < \lambda\}$. Then $d : S \to \mathbb{C}$, $d(z) = e^{-iz}$ is a so-called universal covering map to

$$d(S) = A = A(0, 1, e^\lambda) = \left\{z \in \mathbb{C}\,;\, 1 < |z| < e^\lambda\right\}.$$

We obtain

$$\rho_A(d(z)) = \rho_S(z)|d'(z)| = \rho_L(z) = \frac{\pi}{2\lambda \sin\left(\frac{\pi}{\lambda}\mathrm{Im}(z)\right)}.$$

Let fix z and let $w = d(z) = e^{-iz}$. One such w satisfies $z = i\log(w)$. Now $\mathrm{Im}(z) = w$ and $|d'(z)| = |-ie^{-iz}| = |-iw| = |w|$. We obtain

$$\rho_A(w) = \frac{\pi}{2|w|\lambda \sin\left(\frac{\pi}{\lambda}\log|w|\right)}.$$

Calculating hyperbolic densities for general domains is difficult. It has been found that [4]

$$\rho_{\mathbb{C}\setminus\{0,1\}}(z) = \frac{2\pi}{|z||1-z|}\left(\int_{\mathbb{C}} \frac{dm(\zeta)}{|\zeta||1-\zeta||\zeta - z|}\right)^{-1}.$$

However, this expression has not been very useful for applications. Instead, estimates are needed.

* * *

[4] S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I, No. 413, 1968, 12 pp.

Solution 2.7. (a) Since

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

we have

$$\lim_{\lambda\to\infty}\frac{\pi/\lambda}{2\sin(\frac{\pi}{\lambda}\operatorname{Im}(z))}=\frac{1}{2\operatorname{Im}(z)}.$$

(b) Since

$$\rho_{A(0,1,R)}(z)=\frac{\frac{\pi\log|z|}{\log R}}{2\sin\left(\frac{\pi\log|z|}{\log R}\right)|z|\log|z|},$$

we have for $A(0,1,\infty)=\{1<|z|\}$ that

$$\rho_{A(0,1,\infty)}(z)=\frac{|dz|}{2|z|\log|z|}.$$

(c) By doing a change of variable $z=1/w$, $|dz|=|dw|/|w|^2$, we obtain

$$\rho_{\mathbb{D}^*}(z)=\frac{|dw|}{2|w|\log\frac{1}{|w|}}.$$