Research Seminar in Mathematics 2019
Exercise session: 24.5. at 10–12 in room M305

Exercise 3.1 The chordal and spherical distance of $a, b \in \mathbb{C}$ are defined as

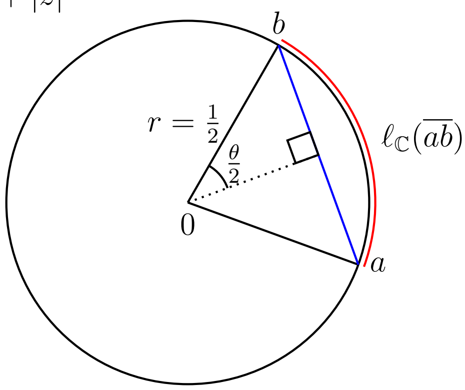
$$[a,b] = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}, \qquad \ell_{\overline{\mathbb{C}}}(\overline{ab}) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1+|z|^2},$$

where the infimum is taken over all piecewise smooth curves joining $a, b \in \mathbb{C}$. Show that

$$[a,b] \leq \ell_{\mathbb{C}}(\overline{ab}) \leq \frac{\pi}{2}[a,b].$$

Hint. Show that

$$\sin x \geq \frac{2}{\pi}x, \quad 0 \leq x \leq \pi.$$



Solution.

Exercise 3.2 (a) Show that

$$\int \frac{dt}{1+t^2} = \arctan t + C. \qquad \text{(substitute: } t = \tan s\text{)}$$

(b) Show that for $a \in (0, \infty)$

$$\int_0^1 \frac{dt}{1+at^2} = \frac{\arctan(\sqrt{a})}{\sqrt{a}}. \qquad \text{(substitute: } t\sqrt{a} = s\text{)}$$

Solution.

Exercise 3.3 Show that for $a, b \in (0, \infty)$

$$\int_0^1 \frac{dt}{1+t^2a+(1-t)^2b} = \frac{\arctan(x)}{x}, \quad \text{where} \quad x = \sqrt{a+b+ab}.$$

Solution.

Hint. The calculation can be done as standard:

(1) show that the roots α_1 and α_2 of the second degree polynomial

$$p(t) = 1 + t^2a + (1-t)^2b$$

are

$$\alpha_1 = \frac{b+ix}{a+b}, \qquad \alpha_1 = \frac{b-ix}{a+b};$$

(2) do the partial fractions decomposition

$$\frac{1}{p(t)} = \frac{A}{t-\alpha_1} + \frac{B}{t-\alpha_2}, \qquad \text{where} \quad A = \frac{1}{2ix} = -B.$$

(3) perform the integration;
(4) simplify the answer by

$$\frac{1}{ix}\frac{1}{2}\log\frac{1+ix}{1-ix} = \frac{\tanh^{-1}(ix)}{ix} = \frac{\arctan(x)}{x}.$$

Exercise 3.4 Show that

$$\frac{\arctan(x)}{x} \leq \frac{\pi}{2}\frac{1}{\sqrt{1+x^2}}, \qquad x \in \mathbb{R}.$$

Hint. Show that

$$h(x) = \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}} - \arctan(x) \geq 0, \quad \text{for} \quad x \in (0,\infty).$$

Solution.

Exercise 3.5 In the punctured disc $\mathbb{D} \setminus \{0\}$, the hyperbolic density and metric are

$$\rho(z) = \frac{1}{|z|\log\frac{1}{|z|}}, \qquad d_{\mathbb{D}\setminus\{0\}}(a,b) = \inf_{\gamma}\int_{\gamma}\rho(\zeta)|d\zeta|,$$

where the infimum is taken over all piecewise smooth γ joining the points $a, b \in \mathbb{D} \setminus \{0\}$.

(a) Show that $\min_{z\in\mathbb{D}\setminus\{0\}} \rho(z) = e$.

(b) Deduce that $d_{\mathbb{D}\setminus\{0\}}(a,b) \geq e|a-b|$ for $a, b \in \mathbb{D} \setminus \{0\}$. Solution.

Exercise 3.6 The hyperbolic density of the triply-punctured sphere $\overline{\mathbb{C}} \setminus \{0,1,\infty\}$ is

$$\rho(z) = \frac{2\pi}{|z||1-z|}\left(\int_{\mathbb{C}}\frac{dm(\zeta)}{|\zeta||1-\zeta||\zeta-z|}\right)^{-1}$$

and its minimum is $\rho(-1)$.

(a) Show that

$$\frac{\pi}{\rho(-1)} = \int_{\mathbb{C}}\frac{dm(\zeta)}{|\zeta||1-\zeta||\zeta+1|} = \int_0^{\infty}\int_0^{2\pi}\frac{1}{|1-r^2e^{i2\theta}|}\,d\theta\,dr. \tag{3.1}$$

(b) Denote the value of the integral (3.1) by I. Find some constants α, β such that $0 < \alpha \leq I \leq \beta < \infty$. (Any constants are enough.)

(c) Can you calculate the value of I? (Probably difficult.)

Solution.

Exercise 3.7 If $f : A \to B$ is conformal and $g = f^{-1}$, then we obtain for the hyperbolic densities

$$\rho_B(z) = \rho_A(g(z))|g'(z)|.$$

Show that if $B \subset A$, then $\rho_B \geq \rho_A$.

Hint: Let $a \in B$ be arbitrary and show $\rho_B(a) \geq \rho_A(a)$ as follows:

- let $f : A \to B$ be conformal such that $f(a) = a$;
- let $\varphi : \mathbb{D} \to A$ be conformal such that $\varphi(0) = a$;
- consider $\varphi^{-1} \circ f \circ \varphi$ and use the Schwarz lemma.

Solution.

Exercise 3.8 Does there exist $\alpha, \beta \in (0, \infty)$ such that

$$\alpha d_{\mathbb{D}\setminus\{0\}}(a,b) \le d_{\overline{\mathbb{C}}\setminus\{0,1,\infty\}}(a,b) \le \beta d_{\mathbb{D}\setminus\{0\}}(a,b),$$

for all $a, b \in \mathbb{D} \setminus \{0\}$? If they exist, find some values.

Solution.

Solutions.

Solution 3.1. Geometric proof. By elementary trigonometry, $[\zeta, \eta] = \sin(\theta/2)$, while $\ell_{\mathbb{C}}(\zeta, \eta) = \theta/2$. We see that

$$\sin x \ge \frac{2}{\pi} x, \quad 0 \le x \le \pi.$$

Namely,

$$h(x) = \sin(x) - \frac{2}{\pi} x$$

satisfies $h(0) = 0$, $h(\pi/2) = 0$ and $h'(0) > 0$. We obtain that

$$\frac{\ell_{\mathbb{C}}(\zeta, \eta)}{[\zeta, \eta]} = \frac{\theta}{2\sin(\theta/2)} \le \frac{\theta}{2\frac{2}{\pi}\frac{\theta}{2}} = \frac{\pi}{2},$$

as desired.

Analytic proof in a special case. Let $\theta = \angle\zeta 0 \eta \le \pi/2$. Set $\gamma(t) = t\zeta + (1-t)\eta$, $0 \le t \le 1$. Then we have

$$\ell_{\mathbb{C}}(\overline{\zeta\eta}) = \int_0^1 \frac{|\gamma'(t)|}{1 + |\gamma(t)|^2}\, dt = |\zeta - \eta| \int_0^1 \frac{dt}{1 + |\gamma(t)|^2}.$$

Since $\angle\zeta 0 \eta < \pi/2$, we have

$$\mathrm{Re}(\zeta\overline{\eta}) \ge 0.$$

Therefore

$$\begin{aligned} |\gamma(t)|^2 &= t^2|\zeta|^2 + (1-t)^2|\eta|^2 + 2t(1-t)\mathrm{Re}(\zeta\overline{\eta}) \\ &\ge t^2|\zeta|^2 + (1-t)^2|\eta|^2, \end{aligned}$$

for $0 \le t \le 1$. Hence, we obtain

$$\ell_{\mathbb{C}}(\overline{\zeta\eta}) \le |\zeta - \eta| \int_0^1 \frac{1}{1 + t^2|\zeta|^2 + (1-t)^2|\eta|^2}\, dt.$$

By Exercise 3.3, we have

$$\int_0^1 \frac{1}{1 + t^2|\zeta|^2 + (1-t)^2|\eta|^2}\, dt = \frac{\arctan(\sqrt{|\zeta|^2 + |\eta|^2 + |\zeta|^2|\eta|^2})}{\sqrt{|\zeta|^2 + |\eta|^2 + |\zeta|^2|\eta|^2}}.$$

By Exercise 3.4, we have

$$\frac{\arctan x}{x} \le \frac{\pi}{2} \frac{1}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}.$$

Hence, we obtain

$$\int_0^1 \frac{1}{1 + t^2|\zeta|^2 + (1-t)^2|\eta|^2}\, dt \le \frac{\pi}{2} \frac{1}{\sqrt{1 + |\zeta|^2}\sqrt{1 + |\eta|^2}}.$$

Thus, we obtain

$$\ell_{\mathbb{C}}(\overline{\zeta\eta}) \le \frac{\pi}{2} \frac{|\zeta - \eta|}{\sqrt{1 + |\zeta|^2}\sqrt{1 + |\eta|^2}} \le \frac{\pi}{2} [\zeta, \eta].$$

* * *

Solution 3.2. (a) Set $t = \tan s$. Now

$$1 + t^2 = 1 + \frac{\sin^2 s}{\cos^2 s} = \frac{1}{\cos^2 s}.$$

Since

$$(\tan s)' = \frac{\cos s(\sin s)' - \sin s(\cos s)'}{\cos^2 s} = \frac{1}{\cos^2 s},$$

we have

$$dt = \frac{1}{\cos^2 s} ds.$$

Hence

$$\int \frac{1}{1+t^2} dt = \int \cos^2 s \frac{1}{\cos^2 s} ds = \int ds = s + C = \arctan(t) + C.$$

(b) Set $\sqrt{a}t = s$. Now, by (a), we obtain

$$\int_0^1 \frac{dt}{1+at^2} = \frac{1}{\sqrt{a}} \int_0^{\sqrt{a}} \frac{ds}{1+s^2} = \frac{1}{\sqrt{a}}(\arctan\sqrt{a} - \arctan 0) = \frac{\arctan\sqrt{a}}{\sqrt{a}}.$$

* * *

Solution 3.3. The denominator

$$h(t) = 1 + ta + (1-t)b = (a+b)t - 2bt + (1+b) = 0$$

if

$$t = \frac{2b \pm \sqrt{4b - 4(a+b)(1+b)}}{2(a+b)} = \frac{b \pm \sqrt{(-1)(a+b+ab)}}{a+b} = \frac{b \pm ix}{a+b},$$

where $x = \sqrt{a+b+ab}$. Let

$$\alpha_1 = \frac{b+ix}{a+b}; \quad \alpha_2 = \frac{b-ix}{a+b}.$$

Now

$$h(t) = (a+b)(t-\alpha_1)(t-\alpha_2) = (a+b)(t + (\alpha_1+\alpha_2)t + \alpha_1\alpha_2)).$$

Hence

$$\alpha_1\alpha_2 = \frac{1+b}{a+b}.$$

Let A, B be constants such that

$$\frac{1}{h(t)} = \frac{A}{t-\alpha_1} + \frac{B}{t-\alpha_2}.$$

Multiplying both sides with $(t-\alpha_1)(t-\alpha_2)$ yields

$$\frac{1}{a+b} = A(t-\alpha_2) + B(t-\alpha_1).$$

By setting $t = \alpha_1$ and $t = \alpha_2$, we see that

$$A = \frac{1}{\alpha_1 - \alpha_2} \frac{1}{a+b} = \frac{1}{2ix} = -B.$$

Hence

$$\int_0^1 \frac{dt}{h(t)} = \frac{1}{2ix}\left[\log\frac{1-\alpha_1}{-\alpha_1} - \log\frac{1-\alpha_2}{-\alpha_2}\right] = \frac{1}{2ix}\log\frac{\alpha_1\alpha_2 - \alpha 2}{\alpha_1\alpha_2 - \alpha_1}.$$

This is equal to

$$\frac{1}{ix}\frac{1}{2}\log\frac{1+ix}{1-ix} = \frac{\tanh^{-1}(ix)}{ix} = \frac{\tan(x)}{x}$$

* * *

Solution 3.4. We see that $h(0) = h(\infty) = 0$. Moreover,

$$h'(x) = \frac{\pi}{2}\frac{1}{(1+x^2)^{1/2}} - \frac{\pi}{2}\frac{x^2}{(1+x^2)^{3/2}} - \frac{1}{1+x^2} = \frac{\pi/2 - \sqrt{1+x^2}}{(1+x^2)^{3/2}} = 0$$

for $x = x_0 \approx 1.211$. Hence, h is increasing on $[0, x_0]$, decreasing on $[x_0, \infty]$, and therefore non-negative.

As a curiosity, $h(x_0) \approx 0.33067$ and therefore

$$\arctan(x) \leq \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}} \leq \arctan(x) + \frac{1}{3},$$

where the exact constant on the right is $x_0 \approx 0.33067$.

* * *

Solution 3.5. Let

$$h(x) = x\log\frac{1}{x} = -x\log(x).$$

Then

$$h(1) = 0 = \lim_{x\to 0} h(x).$$

We have

$$h'(x) = -\log x - 1 = 0$$

for $x = 1/e$. Hence

$$\min_{z\in\mathbb{D}\setminus\{0\}} \rho(z) = \left(\max_{x\in(0,1)} h(x)\right)^{-1} = \left(\frac{1}{e}\right)^{-1} = e.$$

Let $a, b \in \mathbb{D}\setminus\{0\}$. We obtain

$$d_{\mathbb{D}\setminus\{0\}}(a,b) = \inf_{\gamma}\int_{\gamma}\rho(\zeta)|d\zeta| \geq e\int_{\gamma}|d\zeta| \geq e|a-b|.$$

Here

$$\int_{\gamma}|d\zeta| \geq |a-b|$$

because, in the Euclidean sense, shortest path connecting a and b is the straight line.

* * *

Solution 3.6. Let $\zeta = re^{i\theta}$.

(a) Substitute $z = -1$ to the formula to obtain

$$\rho(-1) = \frac{2\pi}{1\cdot 2}\left(\int_{\mathbb{C}}\frac{dm(\zeta)}{|\zeta||1-\zeta||\zeta+1|}\right)^{-1}.$$

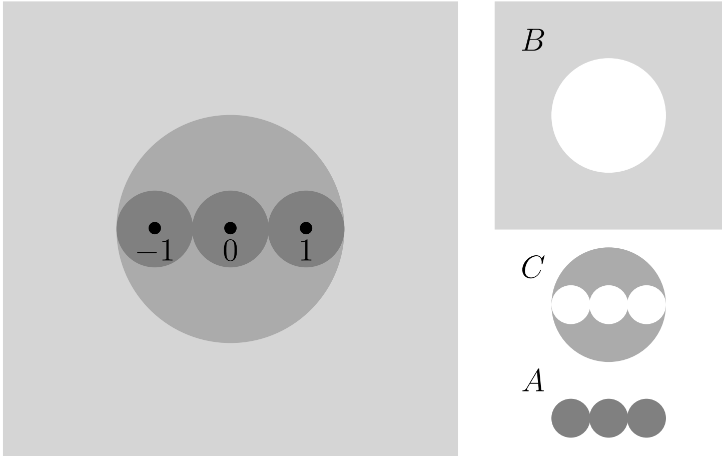


FIGURE 3.1. Dividing $\mathbb{C}$ into three parts.

The Lebesgue area element $dm(\zeta) = r dr d\theta$ and $|1-\zeta||\zeta+1| = |1-\zeta^2| = |1-r^2e^{i2\theta}|$. We obtain

$$\frac{\pi}{\rho(-1)} = \int_{\mathbb{C}} \frac{dm(\zeta)}{|\zeta||1-\zeta||\zeta+1|} = \int_0^\infty \int_0^{2\pi} \frac{1}{|1-r^2e^{i2\theta}|} d\theta dr.$$

(b) **Lower estimate.** We calculate a lower estimate for I. For example,

$$|1-\zeta^2| \le 1 + |\zeta|^2 = 1 + r^2$$

and

$$I \ge \int_0^{2\pi} \int_0^\infty \frac{r dr d\theta}{r(1+r^2)} = 2\pi \left[\arctan(r)\right]_{r=0}^\infty = 2\pi \cdot \frac{\pi}{2} = \pi^2 = \alpha.$$

Upper estimate. We calculate an upper estimate for I. To do this, we divide the plane into three parts, as shown in Figure 3.1.

If $r \le 1/2$, then

$$|1-\zeta^2| \ge 1 - r^2 \ge \frac{3}{4}$$

and therefore

$$\int_{D(0,1/2)} \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} \le \pi \left(\frac{1}{2}\right)^2 \frac{4}{3} \le \frac{\pi}{3}.$$

Let $A = D(0,1/2) \cup D(-1,1/2) \cup D(1,1/2)$. Clearly, $|\zeta - 0||\zeta-1||\zeta-(-1)|$ is relatively smallest near the origin. Therefore

$$\int_A \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} \le 3 \int_{D(0,1/2)} \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} = \pi.$$

Let $B = \mathbb{C} \setminus D(0,3/2)$. For $r > a > 1$, we obtain

$$\frac{1}{|1-\zeta^2|} \le \frac{1}{r^2-1} = \frac{1}{2}\left(\frac{1}{r-1} - \frac{1}{r+1}\right)$$

and

$$\int_{|\zeta|\ge a} \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} \le 2\pi \frac{1}{2}\left[\log \frac{a-1}{a+1}\right]_{r=a}^\infty = \pi \log \frac{a+1}{a-1} = \pi \log 5,$$

when $a = 3/2$. Hence
$$\int_B \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} \leq \pi \log 5.$$

Let $C = D(0,3/2) \setminus A$. For $\zeta \in V$, trivially
$$|\zeta||1-\zeta^2| = |\zeta||1-\zeta||1+\zeta| \geq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}.$$

We obtain a better estimate by noting that
$$2\max(|1-\zeta|,|1+\zeta|) \geq |1-\zeta| + |1+\zeta| \geq |(1-\zeta)+(1+\zeta)| = 2.$$
This implies $\max(|1-\zeta|,|1+\zeta|) \geq 1$ and
$$|\zeta||1-\zeta||1+\zeta| \geq \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{4}.$$
Hence
$$\int_C \frac{dm(\zeta)}{|\zeta||1-\zeta^2|} \leq 4\int_C dm(\zeta) = 4\pi\left(\frac{9}{4}-\frac{3}{4}\right) = 6\pi.$$

In conclusion,
$$I = \int_A + \int_B + \int_C \leq \pi + \pi\log 5 + 6\pi \leq 9\pi = \beta.$$
and
$$0 < 9 \leq \pi^2 \leq I \leq 9\pi \leq 28.3 < \infty.$$

(c) It's not totally clear, how to calculate
$$\int_{\overline{\mathbb{C}}} \frac{dm(\zeta)}{|\zeta||1-\zeta^2|}$$
directly.

* * *

Solution 3.7. The hint is actually inaccurate, since if A is not simply connected, there is no invertible $\varphi : \mathbb{D} \to A$.

Also, using a disc $D(a,r) \subset B \subset A$ is not a good idea.

The solution below will be updated later.

However, let $a \in B \subset A$. Let U be an open disc such that $a \in U \subset B$. We choose $f : U \to U$, $f(z) = z$. Let $\varphi : \mathbb{D} \to U$ such that $\varphi(0) = a$. Let $h = \varphi^{-1} \circ f \circ \varphi$. Then $h(0) = 0$.

We obtain
$$h'(z) = (\varphi^{-1})'(f(\varphi(z))f'(\varphi(z))\varphi'(z)$$
which yields
$$h'(0) = (\varphi^{-1})'(a)f'(a)\varphi'(a) = f'(a).$$

Since $h : \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$, Schwarz lemma implies that
$$|f'(a)| = |h'(0)| \leq 1.$$
Hence $g = f^{-1}$ satisfies
$$|g'(a)| \geq 1.$$

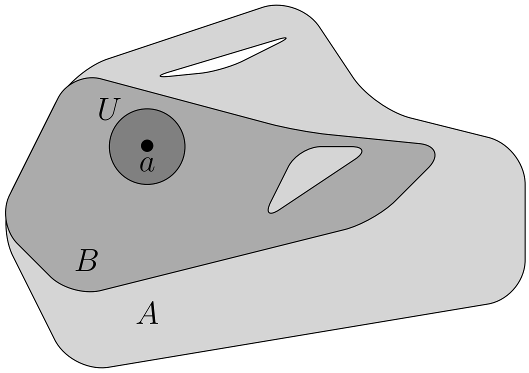


FIGURE 3.2. Slightly irregular domains $B \subset A$.

We obtain

$$\rho_B(a) = \rho_A(g(a))|g'(a)| \geq \rho_A(a).$$

* * *

Solution 3.8. Since $\mathbb{D} \setminus \{0\} \subset \mathbb{C} \setminus \{0,1\}$, we have

$$\rho_{\mathbb{D}\setminus\{0\}} \geq \rho_{\mathbb{C}\setminus\{0,1\}} \quad \text{and} \quad d_{\mathbb{D}\setminus\{0\}} \geq d_{\mathbb{C}\setminus\{0,1\}}.$$

Hence, we can choose $\beta = 1$.

Clearly,

$$d_{\mathbb{C}\setminus\{0,1\}}(-1,-1/2) \in (0,\infty)$$

but

$$d_{\mathbb{D}\setminus\{0\}}(-1,-1/2) = \infty.$$

Hence, no such constant α exists.