Applied Mathematics and Physics in Programming ID00CS50-3004

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Answer to all six questions. In the end, there are some formulas.

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1. Give an example of each type.
   (a) Differential equation with order 3.
   (b) Differential equation which is separable and non-linear.
   (c) Differential equation which is linear and homogeneous.
   (d) Differential equation which is and non-homogeneous.
2. Show that the functions y are solutions to the corresponding differential equations.
   (a) Show that $y = \dfrac{1}{(1-x)^3}$ is not a particular solution for $y' = 2(1-x)y^2$.
   (b) Show that $y = \dfrac{e^{-2x}}{3}$ is a particular solution for $y' + 5y = e^{-2x}$.
3. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -40)$?

4. The solution of

$$y' + p(x)y = q(x)$$

is given by the formula

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Use the formula to solve

$$y' + \frac{5}{x}y = x^2$$

by following the instructions.

(a) Identify $p(x)$ and $q(x)$.

(b) Calculate $\int p(x)dx$. Don't add a constant C yet.

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$

(d) Calculate $\int \mu(x)q(x)dx$.

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

5. Consider the 2π periodic function $f(x)$ which satisfies

$$f(t) = \begin{cases} x, & \text{if } 0 \le x \le \pi \\ -x, & \text{if } -\pi \le x \le 0. \end{cases}$$

Which of the Fourier coefficients a_0, a_1, a_2, b_1 and b_2 are nonzero?

6. Find the discrete Fourier transform of $[2+3i, 3, -5, 7]$. In other words, calculate by hand `fft`$([2+3i, 3, -5, 7])$.

In the following pages, there are some formulas.

Formulas

Differentiation and integration

Differentiation		Integration	
Dx^n	$= nx^{n-1}$	$\int x^n dx$	$= \frac{x^{n+1}}{n+1} + C$
De^x	$= e^x$	$\int e^x dx$	$= e^x + C$
Db^x	$= b^x \ln(b)$	$\int b^x dx$	$= \frac{b^x}{\ln(b)}$
$D\ln(x)$	$= \frac{1}{x}$		
$D\ln\lvert x\rvert$	$= \frac{1}{x}$	$\int \frac{1}{x} dx$	$= \ln\lvert x\rvert + C$
$D\log_a(x)$	$= \frac{1}{x\ln(a)}$		
$D\log_a\lvert x\rvert$	$= \frac{1}{x\ln(a)}$		
$D\sin(x)$	$= \cos(x)$	$\int \cos(x) dx$	$= \sin(x) + C$
$D\cos(x)$	$= -\sin(x)$	$\int \sin(x) dx$	$= -\cos(x) + C$
$D\tan(x)$	$= 1 + \tan^2(x)$	$\int 1 + \tan^2(x) dx$	$= \tan(x) + C$
$Dx\ln(x) - x$	$= \ln(x)$	$\int \ln(x) dx$	$= x\ln(x) - x + C$
$D\arcsin(x)$	$= \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}$	$= \arcsin(x) + C$
$D\arccos(x)$	$= \frac{1}{-\sqrt{1-x^2}}$	$\int \frac{1}{-\sqrt{1-x^2}}$	$= \arccos(x) + C$
$D\arctan(x)$	$= \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}$	$= \arctan(x) + C$
$D\sinh(x)$	$= \cosh(x)$		
$D\cosh(x)$	$= \sinh(x)$		
$D\tanh(x)$	$= \frac{1}{\cosh^2(x)}$		

Differentiation		Integration	
$Df(g(x))$	$= f'(g(x))g'(x)$	$\int f'(g(x))g'(x) dx$	$= f(g(x)) + C$
Special cases			
$D\ln(g(x))$	$= \frac{g'(x)}{g(x)}$	$\int \frac{g'(x)}{g(x)} dx$	$= \ln(g(x)) + C$
$De^{g(x)}$	$= e^{g(x)}g'(x)$	$\int g'(x)e^{g(x)} dx$	$= e^{g(x)} + C$
Dfg	$= f'g + fg'$	$\int f'g dx$	$= fg - \int fg' dx$
$D(f/g)$	$= (gf' - fg')/g^2$		
$Df^{-1}(x)$	$= \frac{1}{f'(f^{-1}(x))}$		

Solution formula

The solution of

$$y' + p(x)y = q(x)$$

is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x)\sin(nx)dx,$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where} \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x)\cos(nx)dx.$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$