

Exam 27.2.2025

Applied Mathematics and Physics in Programming ID00CS50-3003

Answer to all six questions.

1. Match each of the differential equations

(E1) $y''' + 4xy' + 4y = 0$

(E2) $y'' + 4y' + 4y = 0$

(E3) $y' - \tan(x)y = 0$

(E4) $y'' + x \sin(y) = y$

(E5) $y'' + xy' + 4y = x^2$

with one of the following properties. (One equation for one property. No need to justify the answer.)

- (a) Which equation is separable?
- (b) Which equation is of order 3?
- (c) Which equation has constant coefficients?
- (d) Which equation is linear and non-homogeneous?
- (e) Which equation is nonlinear?

2. Match each of the differential equations

(F1) $my'' = -mg$

(F2) $y' = -ay$

(F3) $my'' + ky = 0$

(F4) $my'' + Ry' + ky = 0$

(F5) $y'' + \sin(y) = 0$

with one of the following physics phenomena. (One equation for one phenomenon. No need to justify the answer.)

- (a) Which equation is about radioactive decay?
- (b) Which equation is about pendulum?
- (c) Which equation is about free fall?
- (d) Which equation is about harmonic oscillator (mass and spring) without damping?
- (e) Which equation is about harmonic oscillator with damping?

3. Find the general solution of $y' = 6x + 2$.

Find the particular solution which satisfies the initial condition $y(0) = 3$. For this particular solution, find $y(2)$.

4. A mass ($m = 1 \text{ kg}$) lies on a frictionless rail. The mass is connected to the end of the rail with a spring ($k = 25 \text{ kg/s}^2$). The position of the mass at time t is described by the function $x(t) = A \cos(\omega t) + B \sin(\omega t)$.

A person pulls the mass 0.25 m from the neutral position. At the moment $t = 0$, we have $x(0) = 0.25$ and $x'(0) = 0$. The person releases the mass and the mass will oscillate back and forth.

Tasks.

- (a) Express ω in terms of k and m .
- (b) When does the mass for the first time return to the starting position?
- (c) What is the maximum velocity of the mass?

5. Consider the equation

$$y' - 2 \tan(x)y = x.$$

Solve it, for example, by following the instructions.

- (a) Identify $p(x)$ and $q(x)$.
- (b) Calculate $\int p(x)dx$. Don't add a constant C yet.
- (c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$
- (d) Calculate $\int \mu(x)q(x)dx$.
- (e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

6. Consider the 2π periodic function f which satisfies $f(x) = |x|$ for $-\pi < x < \pi$.

Which of the Fourier coefficients a_0, a_1, a_2, b_1, b_2 are zero?

Formulas

Differentiation and integration

Differentiation

$$Dx^n = nx^{n-1}$$

$$De^x = e^x$$

$$Db^x = b^x \ln(b)$$

$$D \ln(x) = \frac{1}{x}$$

$$D \ln |x| = \frac{1}{x}$$

$$D \log_a(x) = \frac{1}{x \ln(a)}$$

$$D \log_a |x| = \frac{1}{x \ln(a)}$$

$$D \sin(x) = \cos(x)$$

$$D \cos(x) = -\sin(x)$$

$$D \tan(x) = 1 + \tan^2(x)$$

$$Dx \ln(x) - x = \ln(x)$$

$$D \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$D \arccos(x) = \frac{1}{-\sqrt{1-x^2}}$$

$$D \arctan(x) = \frac{1}{1+x^2}$$

$$D \sinh(x) = \cosh(x)$$

$$D \cosh(x) = \sinh(x)$$

$$D \tanh(x) = \frac{1}{\cosh^2(x)}$$

Integration

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C$$

$$\int b^x dx = \frac{b^x}{\ln(b)}$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int 1 + \tan^2(x) dx = \tan(x) + C$$

$$\int \ln(x) dx = x \ln(x) - x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int \frac{1}{-\sqrt{1-x^2}} = \arccos(x) + C$$

$$\int \frac{1}{1+x^2} = \arctan(x) + C$$

Differentiation

$$Df(g(x)) = f'(g(x))g'(x)$$

Special cases

$$D \ln(g(x)) = \frac{g'(x)}{g(x)}$$

$$De^{g(x)} = e^{g(x)}g'(x)$$

$$Dfg = f'g + fg'$$

$$D(f/g) = (gf' - fg')/g^2$$

Integration

$$\int f(g(x))g'(x) dx = f(g(x)) + C$$

$$\int \frac{g'(x)}{g(x)} dx = \ln(g(x)) + C$$

$$\int g'(x)e^{g(x)} dx = e^{g(x)} + C$$

$$\int f'g dx = fg - \int fg' dx$$

Differential equations

Second order linear ODE with constant coefficients

- ODE $y'' + by' + cy = 0$
- Characteristic equation $r^2 + br + c = 0$

Cases

- $r_1, r_2 \in \mathbb{R}$ solution

$$y(x) = A \exp(r_1 x) + B \exp(r_2 x)$$

- $r_1 = r_2 = r$ solution

$$y(x) = A \exp(rx) + Bx \exp(rx)$$

- $r_1 = a + bi$ solution

$$y(x) = \exp(ax)(A \cos(bx) + B \sin(bx))$$

Integrable ODE

The solution of

$$y' = q(x)$$

is $y(x) = \int q(x) dx$

Separable ODE

If you can arrange the equation as

$$a(y)dy = b(x)dx,$$

then you can integrate to obtain

$$\int a(y)dy = \int b(x)dx.$$

First order linear ODE

The solution of

$$y' + p(x)y = q(x)$$

is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Trigonometry

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$2 \sin(a) \sin(b) = \cos(a - b) - \cos(a + b)$$

$$2 \sin(a) \cos(b) = \sin(a - b) + \sin(a + b)$$

$$2 \cos(a) \cos(b) = \cos(a - b) + \cos(a + b)$$

Fourier series

If f is periodic with period 2π and f , f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$