Exam 27.2.2025 Name:

Applied Mathematics and Physics in Programming ID00CS50-3003

Answer to all six questions.

1. Match each of the differential equations

   (E1) $y''' + 4xy' + 4y = 0$

   (E2) $y'' + 4y' + 4y = 0$

   (E3) $y' - \tan(x)y = 0$

   (E4) $y'' + x\sin(y) = y$

   (E5) $y'' + xy' + 4y = x^2$

   with one of the following properties. (One equation for one property. No need to justify the answer.)

   (a) Which equation is separable?

   (b) Which equation is of order 3?

   (c) Which equation has constant coefficients?

   (d) Which equation is linear and non-homogeneous?

   (e) Which equation is nonlinear?

2. Match each of the differential equations

   (F1) $my'' = -mg$

   (F2) $y' = -ay$

   (F3) $my'' + ky = 0$

   (F4) $my'' + Ry' + ky = 0$

   (F5) $y'' + \sin(y) = 0$

   with one of the following physics phenomena. (One equation for one phenomenon. No need to justify the answer.)

   (a) Which equation is about radioactive decay?

   (b) Which equation is about pendulum?

   (c) Which equation is about free fall?

   (d) Which equation is about harmonic oscillator (mass and spring) without damping?

   (e) Which equation is about harmonic oscillator with damping?

3. Find the general solution of $y' = 6x + 2$.

   Find the particular solution which satisfies the initial condition $y(0) = 3$. For this particular solution, find $y(2)$.

4. A mass ($m = 1\ kg$) lies on a frictionless rail. The mass is connected to the end of the rail with a spring ($k = 25\ kg/s^2$). The position of the mass at time t is described by the function $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

   A person pulls the mass 0.25 m from the neutral position. At the moment $t = 0$, we have $x(0) = 0.25$ and $x'(0) = 0$. The person releases the mass and the mass will oscillate back and forth.

   Tasks.

   (a) Express ω in terms of k and m.

   (b) When does the mass for the first time return to the starting position?

   (c) What is the maximum velocity of the mass?

5. Consider the equation

$$y' - 2\tan(x)y = x.$$

   Solve it, for example, by following the instructions.

   (a) Identify $p(x)$ and $q(x)$.

   (b) Calculate $\int p(x)dx$. Don't add a constant C yet.

   (c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$

   (d) Calculate $\int \mu(x)q(x)dx$.

   (e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

6. Consider the 2π periodic function f which satisfies $f(x) = |x|$ for $-\pi < x < \pi$.

   Which of the Fourier coefficients a_0, a_1, a_2, b_1, b_2 are zero?

Formulas

Differentiation and integration

Differentiation	**Integration**
$Dx^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$De^x = e^x$	$\int e^x dx = e^x + C$
$Db^x = b^x \ln(b)$	$\int b^x dx = \frac{b^x}{\ln(b)}$
$D\ln(x) = \frac{1}{x}$	
$D\ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$D\log_a(x) = \frac{1}{x\ln(a)}$	
$D\log_a\lvert x\rvert = \frac{1}{x\ln(a)}$	
$D\sin(x) = \cos(x)$	$\int \cos(x) dx = \sin(x) + C$
$D\cos(x) = -\sin(x)$	$\int \sin(x) dx = -\cos(x) + C$
$D\tan(x) = 1 + \tan^2(x)$	$\int 1 + \tan^2(x) dx = \tan(x) + C$
$Dx\ln(x) - x = \ln(x)$	$\int \ln(x) dx = x\ln(x) - x + C$
$D\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C$
$D\arccos(x) = \frac{1}{-\sqrt{1-x^2}}$	$\int \frac{1}{-\sqrt{1-x^2}} = \arccos(x) + C$
$D\arctan(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} = \arctan(x) + C$
$D\sinh(x) = \cosh(x)$	
$D\cosh(x) = \sinh(x)$	
$D\tanh(x) = \frac{1}{\cosh^2(x)}$	

Differentiation	**Integration**
$Df(g(x)) = f'(g(x))g'(x)$	$\int f'(g(x))g'(x)dx = f(g(x)) + C$
Special cases	
$D\ln(g(x)) = \frac{g'(x)}{g(x)}$	$\int \frac{g'(x)}{g(x)} dx = \ln(g(x)) + C$
$De^{g(x)} = e^{g(x)}g'(x)$	$\int g'(x)e^{g(x)} dx = e^{g(x)} + C$
$Dfg = f'g + fg'$	$\int f'g dx = fg - \int fg' dx$
$D(f/g) = (gf' - fg')/g^2$	

Differential equations

Second order linear ODE with constant coefficients

- ODE $y'' + by' + cy = 0$
- Characteristic equation $r^2 + br + c = 0$

Cases

- $r_1, r_2 \in \mathbb{R}$ solution
$$y(x) = A\exp(r_1 x) + B\exp(r_2 x)$$
- $r_1 = r_2 = r$ solution
$$y(x) = A\exp(rx) + Bx\exp(rx)$$
- $r_1 = a + bi$ solution
$$y(x) = \exp(ax)(A\cos(bx) + B\sin(bx))$$

Integrable ODE

The solution of
$$y' = q(x)$$
is $y(x) = \int q(x)dx$

Separable ODE

If you can arrange the equation as
$$a(y)dy = b(x)dx,$$
then you can integrate to obtain
$$\int a(y)dy = \int b(x)dx.$$

First order linear ODE

The solution of
$$y' + p(x)y = q(x)$$
is
$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Trigonometry

$$\begin{aligned}
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b) \\
2\sin(a)\cos(b) &= \sin(a-b) + \sin(a+b) \\
2\cos(a)\cos(b) &= \cos(a-b) + \cos(a+b)
\end{aligned}$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx
\end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$