Exam 1.4.2025

Applied Mathematics and Physics in Programming ID00CS50-3003

Answer to all six questions.

1. Match each of the differential equations

   (E1) $y''' + 4xy' + 4y = 0$

   (E2) $y'' + 4y' + 4y = 0$

   (E3) $y' - \tan(x)y = 0$

   (E4) $y'' + x\sin(y) = y$

   (E5) $y'' + xy' + 4y = x^2$

   with one of the following properties. (One equation for one property. No need to justify the answer.)

   (a) Which equation is separable?

   (b) Which equation is of order 3?

   (c) Which equation has constant coefficients?

   (d) Which equation is linear and non-homogeneous?

   (e) Which equation is nonlinear?

2. Match each of the differential equations

   (F1) $my'' = -mg$

   (F2) $y' = -ay$

   (F3) $my'' + ky = 0$

   (F4) $my'' + Ry' + ky = 0$

   (F5) $y'' + \sin(y) = 0$

   with one of the following physics phenomena. (One equation for one phenomenon. No need to justify the answer.)

   (a) Which equation is about radioactive decay?

   (b) Which equation is about pendulum?

   (c) Which equation is about free fall?

   (d) Which equation is about harmonic oscillator (mass and spring) without damping?

   (e) Which equation is about harmonic oscillator with damping?

3. Find the general solution of $y' = 6x + 2$.

   Find the particular solution which satisfies the initial condition $y(0) = 19$.

   For this particular solution, find $y(2)$.

4. An amount of certain radioactive isotope leaks to an area. The initial radioactivity is measured. After three years, the radioactivity has halved.

   Tasks.

   (a) Which differential equation describes the phenomenon?

   (b) After how many years is the radioactivity 0.1 % of the initial radioactivity?

5. Consider the equation
$$y' - \tan(x)y = x.$$

   Solve it, for example, by following the instructions.

   (a) Identify $p(x)$ and $q(x)$.

   (b) Calculate $\int p(x)dx$. Don't add a constant C yet.

   (c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$

   (d) Calculate $\int \mu(x)q(x)dx$.

   (e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

6. Consider a 2π periodic function f which is both even and odd.

   Can you give an example of such a function?

   Which of the Fourier coefficients a_0, a_1, a_2, b_1, b_2 are zero?

Formulas

Differentiation and integration

<table>
<tr><td colspan="2" align="center">Differentiation</td><td colspan="2" align="center">Integration</td></tr>
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$$
\begin{aligned}
Dx^n &= nx^{n-1} \\
De^x &= e^x \\
Db^x &= b^x \ln(b) \\
D\ln(x) &= \tfrac{1}{x} \\
D\ln|x| &= \tfrac{1}{x} \\
D\log_a(x) &= \tfrac{1}{x\ln(a)} \\
D\log_a|x| &= \tfrac{1}{x\ln(a)} \\
D\sin(x) &= \cos(x) \\
D\cos(x) &= -\sin(x) \\
D\tan(x) &= 1+\tan^2(x) \\
Dx\ln(x)-x &= \ln(x)
\end{aligned}
$$

$$
\begin{aligned}
\int x^n dx &= \tfrac{x^{n+1}}{n+1}+C \\
\int e^x dx &= e^x + C \\
\int b^x dx &= \tfrac{b^x}{\ln(b)} \\
\\
\int \tfrac{1}{x} dx &= \ln|x| + C \\
\\
\\
\int \cos(x)dx &= \sin(x)+C \\
\int \sin(x)dx &= -\cos(x)+C \\
\int 1+\tan^2(x)dx &= \tan(x)+C \\
\int \ln(x)dx &= x\ln(x)-x+C
\end{aligned}
$$

$$
\begin{aligned}
D\arcsin(x) &= \tfrac{1}{\sqrt{1-x^2}} \\
D\arccos(x) &= \tfrac{1}{-\sqrt{1-x^2}} \\
D\arctan(x) &= \tfrac{1}{1+x^2} \\
D\sinh(x) &= \cosh(x) \\
D\cosh(x) &= \sinh(x) \\
D\tanh(x) &= \tfrac{1}{\cosh^2(x)}
\end{aligned}
$$

$$
\begin{aligned}
\int \tfrac{1}{\sqrt{1-x^2}} &= \arcsin(x)+C \\
\int \tfrac{1}{-\sqrt{1-x^2}} &= \arccos(x)+C \\
\int \tfrac{1}{1+x^2} &= \arctan(x)+C
\end{aligned}
$$

<table>
<tr><td align="center">Differentiation</td><td align="center">Integration</td></tr>
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$$
Df(g(x)) = f'(g(x))g'(x) \qquad \int f'(g(x))g'(x)dx = f(g(x))+C
$$

Special cases

$$
\begin{aligned}
D\ln(g(x)) &= \tfrac{g'(x)}{g(x)} \\
De^{g(x)} &= e^{g(x)}g'(x)
\end{aligned}
\qquad
\begin{aligned}
\int \tfrac{g'(x)}{g(x)}dx &= \ln(g(x))+C \\
\int g'(x)e^{g(x)}dx &= e^{g(x)}+C
\end{aligned}
$$

$$
\begin{aligned}
Dfg &= f'g+fg' \\
D(f/g) &= (gf'-fg')/g^2
\end{aligned}
\qquad
\int f'g\,dx = fg - \int fg'dx
$$

Differential equations

Second order linear ODE with constant coefficients

- ODE $y'' + by' + cy = 0$

- Characteristic equation $r^2 + br + c = 0$

Cases

- $r_1, r_2 \in \mathbb{R}$ solution
$$y(x) = A\exp(r_1 x) + B\exp(r_2 x)$$

- $r_1 = r_2 = r$ solution
$$y(x) = A\exp(rx) + Bx\exp(rx)$$

- $r_1 = a + bi$ solution
$$y(x) = \exp(ax)(A\cos(bx) + B\sin(bx))$$

Integrable ODE

The solution of
$$y' = q(x)$$
is $y(x) = \int q(x)dx$

Separable ODE

If you can arrange the equation as
$$a(y)dy = b(x)dx,$$
then you can integrate to obtain
$$\int a(y)dy = \int b(x)dx.$$

First order linear ODE

The solution of
$$y' + p(x)y = q(x)$$
is
$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Trigonometry

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a-b) + \sin(a+b)$$
$$2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 = x_0 + x_1 \\ y_1 = x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 = \frac{1}{2}(x_0 + x_1) \\ y_1 = \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 = x_0 + x_1 + x_2 + x_3 \\ y_1 = x_0 - ix_1 - x_2 + ix_3 \\ y_2 = x_0 - x_1 + x_2 - x_3 \\ y_3 = x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 = \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 = \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 = \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 = \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$