Exam 27.2.2025, solutions Name:

Applied Mathematics and Physics in Programming ID00CS50-3003

Answer to all six questions.

1. Match each of the differential equations

   (E1) $y''' + 4xy' + 4y = 0$

   (E2) $y'' + 4y' + 4y = 0$

   (E3) $y' - \tan(x)y = 0$

   (E4) $y'' + x\sin(y) = y$

   (E5) $y'' + xy' + 4y = x^2$

   with one of the following properties. (One equation for one property. No need to justify the answer.) Note. You get 1p from each part.

   (a) Which equation is separable? Solution. (E3), equation (E3) does not satisfy any of the properties (b)-(e)

   (b) Which equation is of order 3? Solution. (E1), only equation with y'''

   (c) Which equation has constant coefficients? Solution. (E2)

   (d) Which equation is linear and non-homogeneous? Solution. (E5)

   (e) Which equation is nonlinear? Solution. (E4)

2. Match each of the differential equations

   (F1) $my'' = -mg$

   (F2) $y' = -ay$

   (F3) $my'' + ky = 0$

   (F4) $my'' + Ry' + ky = 0$

   (F5) $y'' + \sin(y) = 0$

   with one of the following physics phenomena. (One equation for one phenomenon. No need to justify the answer.) Note. You get 1p from each part.

   (a) Which equation is about radioactive decay? Solution. (F2), it has no oscillating solutions

   (b) Which equation is about pendulum? Solution. (F5), this was the nonlinear equation which we could not solve by hand

   (c) Which equation is about free fall? Solution. (F1), g is the gravity constant

   (d) Which equation is about harmonic oscillator (mass and spring) without damping? Solution. (F3), its solution is $y(x) = A\sin(\omega t + \varphi)$ which oscillates forever.

   (e) Which equation is about harmonic oscillator with damping? Solution. (F4)

3. Find the general solution of $y' = 6x + 2$. Solution. $y(x) = 3x^2 + 2x + C$

   Find the particular solution which satisfies the initial condition $y(0) = 3$. Solution. $y(x) = 3x^2 + 2x + 3$

   For this particular solution, find $y(2)$. Solution. 19

   Note. The problem is graded as whole. Depending on the explanation / Points are given with the idea $1.6 + 1.6 + 1.6$ and then rounded up.

4. A mass ($m = 1\ kg$) lies on a frictionless rail. The mass is connected to the end of the rail with a spring ($k = 25\ kg/s^2$). The position of the mass at time t is described by the function $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

   A person pulls the mass $0.25\ m$ from the neutral position. At the moment $t = 0$, we have $x(0) = 0.25$ and $x'(0) = 0$. The person releases the mass and the mass will oscillate back and forth.

   Solution. By the initial conditions, we have

$$x(0) = A \cdot 1 + B \cdot 0 = A = 0.25$$

   and since $x'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ we have also

$$x'(0) = -A\omega \cdot 0 + B\omega \cdot 1 = B = 0.$$

   The particular solution which satisfies the initial conditions is $x(t) = 0.25\cos(\omega t)$.

   From problem 2, we copy the differential equation $mx'' + kx = 0$. Our particular solution has second derivative $x''(t) = -0.25\omega^2 x(t)$. We have

$$mx'' + kx = 0.25(-m\omega^2 + k)x(t) = 0$$

   if $-m\omega^2 + k = 0$ which yields $\omega = \sqrt{k/m} = \sqrt{1/25} = \frac{1}{5}$.

   Tasks.

   (a) Express ω in terms of k and m. Solution. $\omega = \sqrt{k/m}$

   (b) When does the mass for the first time return to the starting position? Solution. We have $x(t) = 0.25\cos(\omega t) = 0.25$ for some $t > 0$. It must be $\cos(\omega t) = 1$. Hence, $\omega t = 2\pi n$ for $n = 1$. We have $\frac{1}{5}t = 2\pi$ implying $t = 10\pi$.

   (c) What is the maximum velocity of the mass? Solution. Because there is no damping, the energy is preserved. In the neutral position, the system has only kinetic energy. Assuming that the spring has no mass, the kinetic energy is only in the mass. The kinetic energy $\frac{1}{2}mv^2$ is equal to the potential energy $\frac{1}{2}kx^2$ in the beginning. We get

$$mv^2 = kx(0)^2$$

   Implying $v = \sqrt{\frac{k}{m}}x(0) = \omega x(0) = \frac{1}{5}0.25m = 0.05\ m/s$.

   Note. The problem is graded as whole. Depending on the explanation / Points are given with the idea $1.6 + 1.6 + 1.6$ and then rounded up.

5. Consider the equation

$$y' - 2\tan(x)y = x.$$

Solve it, for example, by following the instructions. Note. You get 1p from each part.

(a) Identify $p(x)$ and $q(x)$. Solution. $p(x) = -2\tan(x)$ and $q(x) = x$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Solution. $\int p(x)dx = \ln(\cos^2(x))$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Solution. $\mu(x) = \cos^2(x)$ and $\frac{1}{\mu(x)} = \frac{1}{\cos^2(x)}$

(d) Calculate $\int \mu(x)q(x)dx$. Solution. We need to integrate

$$\int x\cos^2(x)dx$$

Choosing $x = a = b$ in

$$2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$$

we obtain $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and need to integrate

$$I = \frac{1}{2}\int x(1+\cos(2x))dx.$$

We have

$$I = \frac{x^2}{4} + \frac{1}{2}\int x\cos(2x)dx.$$

Partial integration gives

$$\int x\cdot\cos(2x)dx = x\cdot\sin(2x)/2 - \int 1\cdot\sin(2x)/2dx = x\sin(2x)/2 + \cos(2x)/4.$$

We have

$$\int x\cos^2(x)dx = \frac{2x^2 + 2x\sin(2x) + \cos(2x)}{8}.$$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx$. Solution. $y(x) = \frac{C}{\cos^2(x)} + \frac{2x^2+2x\sin(2x)+\cos(2x)}{8\cos^2(x)}$

6. Consider the 2π periodic function f which satisfies $f(x) = |x|$ for $-\pi < x < \pi$.

Which of the Fourier coefficients a_0, a_1, a_2, b_1, b_2 are zero? Note. You get 1p from each a_1, a_2, b_1, b_2. Extra points for smart approach.

Solution. Because $f(-x) = |-x| = |x| = f(x)$, the function f is even which implies that $b_1 = b_2 = 0$.

We have

$$a_0 = \frac{2}{\pi}\int_0^\pi xdx = 2\pi \neq 0.$$

Also

$$a_1 = \frac{2}{\pi}\int_0^\pi x\cos(x)dx = \frac{2}{\pi}\left[x\sin(x) + \cos(x)\right]_{x=0}^{x=\pi} = \frac{2}{\pi}[\cos(\pi) - \cos(0)] = -\frac{2}{\pi} \neq 0.$$

Moreover,

$$a_2 = \frac{2}{\pi}\int_0^\pi x\cos(2x)dx = \frac{2}{\pi}\left[\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right]_{x=0}^{x=\pi} = \frac{1}{2\pi}[\cos(2\pi) - \cos(0)] = 0.$$

In conclusion, we have that $a_2 = b_1 = b_2 = 0$ and $a_0 \neq 0$ and $a_1 \neq 0$.

Formulas

Differentiation and integration

Differentiation	**Integration**
$Dx^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$De^x = e^x$	$\int e^x dx = e^x + C$
$Db^x = b^x \ln(b)$	$\int b^x dx = \frac{b^x}{\ln(b)}$
$D\ln(x) = \frac{1}{x}$	
$D\ln\|x\| = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$D\log_a(x) = \frac{1}{x\ln(a)}$	
$D\log_a\|x\| = \frac{1}{x\ln(a)}$	
$D\sin(x) = \cos(x)$	$\int \cos(x)dx = \sin(x) + C$
$D\cos(x) = -\sin(x)$	$\int \sin(x)dx = -\cos(x) + C$
$D\tan(x) = 1 + \tan^2(x)$	$\int 1 + \tan^2(x)dx = \tan(x) + C$
$Dx\ln(x) - x = \ln(x)$	$\int \ln(x)dx = x\ln(x) - x + C$
$D\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C$
$D\arccos(x) = \frac{1}{-\sqrt{1-x^2}}$	$\int \frac{1}{-\sqrt{1-x^2}} = \arccos(x) + C$
$D\arctan(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} = \arctan(x) + C$
$D\sinh(x) = \cosh(x)$	
$D\cosh(x) = \sinh(x)$	
$D\tanh(x) = \frac{1}{\cosh^2(x)}$	

Differentiation	**Integration**
$Df(g(x)) = f'(g(x))g'(x)$	$\int f'(g(x))g'(x)dx = f(g(x)) + C$
Special cases	
$D\ln(g(x)) = \frac{g'(x)}{g(x)}$	$\int \frac{g'(x)}{g(x)} dx = \ln(g(x)) + C$
$De^{g(x)} = e^{g(x)}g'(x)$	$\int g'(x)e^{g(x)}dx = e^{g(x)} + C$
$Dfg = f'g + fg'$	$\int f'g dx = fg - \int fg' dx$
$D(f/g) = (gf' - fg')/g^2$	

Differential equations

Second order linear ODE with constant coefficients

- ODE $y'' + by' + cy = 0$
- Characteristic equation $r^2 + br + c = 0$

Cases

- $r_1, r_2 \in \mathbb{R}$ solution
$$y(x) = A\exp(r_1 x) + B\exp(r_2 x)$$
- $r_1 = r_2 = r$ solution
$$y(x) = A\exp(rx) + Bx\exp(rx)$$
- $r_1 = a + bi$ solution
$$y(x) = \exp(ax)(A\cos(bx) + B\sin(bx))$$

Integrable ODE

The solution of
$$y' = q(x)$$
is $y(x) = \int q(x)dx$

Separable ODE

If you can arrange the equation as
$$a(y)dy = b(x)dx,$$
then you can integrate to obtain
$$\int a(y)dy = \int b(x)dx.$$

First order linear ODE

The solution of
$$y' + p(x)y = q(x)$$
is
$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Trigonometry

$$\begin{aligned}
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b) \\
2\sin(a)\cos(b) &= \sin(a-b) + \sin(a+b) \\
2\cos(a)\cos(b) &= \cos(a-b) + \cos(a+b)
\end{aligned}$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$